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A Concentration Theorem of $(\mathbf{R}, p)$-anders on Hadamard Manifolds

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Abstract: In this note, we prove a concentration theorem of $(\mathbf{R}, p)$-anders. As a simple corollary, one can prove that (X, p)-anders do not admit coarse embeddings into Hadamard manifolds with bounded sectional curvatures.

Key words: expander, $(\mathbf{R}, p)$-ander, concentration theorem, coarse embedding

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Coarse Baum-Connes Conjecture is one of the most important conjectures in the non-commutative geometry. It provides an algorithm of calculation of indices of certain differential operators. For example, it implies the zero-in-the-specturm conjecture stating that the Laplacian operator acting on the space of all L^2-forms of a uniformly contractible Riemannian manifold has zero in its spectrum (see [1]). The celebrated work of Yu[2] asserts that any metric spaces which can be coarsely embedded into Hilbert space satisfy the Coarse Baum-Connes Conjecture. Later Yu and Kasparov[1] prove that any metric spaces which can be coarsely embedded into a uniformly convex Banach space satisfy the injectivity of Coarse Baum-Connes Conjecture. Recently, Chen *et al.*[3] prove the maximal Coarse Baum-Connes Conjecture for spaces which admit a fibred coarse embedding into Hilbert space. On the other hand, now it is well-known that expanders do not admit a coarse embedding into Hilbert spaces and there exist expanders which do not admit a coarse embedding into uniformly convex Banach space (see [4]). Gong *et al.*[5] prove the Coarse Geometric Novikov Conjecture for a large class of expanders, especially the expanders in [4]. The Coarse Novikov Conjecture, or the maximal Coarse Novikov Conjecture, is known to be true for more classes of expanders by [6]–[9]. Moreover, Oyono-Oyono and Yu[8] also prove isomorphism of the maximal version of the coarse assembly map. These facts make the study of expanders

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extremely important in the non-commutative geometry.

The non-coarse embeddability of expanders into Hilbert space is triggered by a concentration theorem of expanders. In [10], a concentration theorem of expanders for Banach space whose unit balls are uniformly embeddable into Hilbert space is proved. In [11], the author proves a concentration theorem of expanders for Hadamard manifolds.

Recently, Mimura[12] introduces Banach spectral gap and (X, p)-anders. Here we prove a concentration theorem for $(\mathbf{R}, p)$-anders for Hadamard manifolds.

First, let us recall basics of (X, p)-anders (see [12]). Let $G = (V, E)$ be a finite graph with the vertex set V and the edge set E. Denote the cardinality of V and E by $|V|$ and $|E|$, respectively. The degree of a vertex v is the number of edges incident to v. The maximum degree of G, denoted by $\Delta(G)$, is the maximum degree of its vertices. We equip G with the path metric and regard as a metric space.

Definition 1[12] *Let (X, p) be a pair of a Banach space X and an exponent $p > 0$.*

(I) *The Banach spectral gap is defined as follows: the (X, p)-spectral gap of G, written as $\lambda_1(G; X, p)$, is*

$$\lambda_1(G; X, p) = \inf_f \frac{1}{2} \cdot \frac{\displaystyle\sum_{v\in V} \sum_{e=(v,w)\in E} \|f(w) - f(v)\|^p}{\displaystyle\sum_{v\in V} \|f(v) - m(f)\|^p}.$$

Here $f : V \to X$ runs over all nonconstant maps and $m(f) = \dfrac{1}{|V|}\displaystyle\sum_{v\in V} f(v)$;

(II) *A sequence of finite connected graphs $\{G_n\}$ is called (X, p)-anders if the following three conditions are satisfied:*

(1) $\sup\limits_n \Delta(G_n) < \infty$;

(2) $\lim\limits_{n\to\infty} |V_n| = \infty$;

(3) $\inf\limits_n \lambda_1(G_n; X, p) > 0$.

In this definition, we do not require the orientation of graphs. Every edges are counted twice. Hence there is "$\frac{1}{2}$" in the definition. As mentioned in [12], $(\mathbf{R}, 2)$-anders are expanders in usual sense.

A graph is called regular if every vertex has the same degree. A graph is called simple if there is no edge connecting a vertex to itself. From now on, we focus on regular and simple graphs.

Since $\mathbf{R}$ can be isometrically embedded into any non-trivial Banach space X, we have

$$\lambda_1(G; \mathbf{R}, p) \geq \lambda_1(G; X, p).$$

Hence (X, p)-anders are $(\mathbf{R}, p)$-anders.

In the following calculations, we repeatedly use Jensen's inequality

$$\phi\left(\frac{1}{n}\sum_{i=1}^n x_i\right) \leq \frac{1}{n}\sum_{i=1}^n \phi(x_i)$$

for convex function ϕ. Norm functions and x^p with $p \geq 1$ are convex functions.