INTERNATIONAL JOURNAL OF
NUMERICAL ANALYSIS AND MODELING
Volume 18, Number 2, Pages 165–189

ENERGY AND ENSTROPHY STUDY OF THE TIME RELAXATION MODEL

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Abstract. This paper presents an investigation of the convergence of an Oseen type of problem for a fluid model known as Time Relaxation Model, TRM. Furthermore conservation of energy and enstrophy of TRM are presented along with a dimensional analysis study of these properties. The paper contains two types of numerical experiments, one of which is based on finite element discretization that employs the Oseen type of problem for dealing with the nonlinearity of the TRM. The second type is the spectral related tests for the study of TRM dimensional analysis. Our numerical experiments confirm the theoretical findings.

Key words. Time relaxation, finite element, spectral method, energy, enstrophy.

1. Introduction

All the turbulent fluid models aim to create a balance between an accurate description of the fluid flow and the model resolution so that the simulation can be performed within a realistic time limits while the capacity set by today's high performing computers is also taken into consideration, [13, 16, 27]. This is due to the fact that at high Reynolds numbers the fluid's velocity $\mathbf{u}$ has many small spatial scales, which becomes computationally less feasible by standard NSE. Ultimately a very fine mesh is needed to capture all the small scales [1, 12, 11]. In this context, high order fluid models are of interest, since they deliver a more accurate simulation at low model resolution. However, generally speaking, all such models run the risk of altering the largest structures of the flow that contain most of the flow's energy and are responsible for most of the mixing and flow's momentum transport [21].

In the present paper the numerical properties of TRM are investigated including the conservative and spectral behavior in three-dimensional incompressible flows. This fluid model was first introduced by Stolz, Adams and Kleiser [28, 29]. The truncation scale analysis of TRM is shown in [21]. Further numerical studies and computational accuracy of this model can be found in [9, 11, 21, 24]. Parameter sensitivity analysis of TRM was performed in [25, 8].

The governing equations for TRM are formulated as

$$
\begin{aligned}
\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p + \chi(\mathbf{u}-\bar{\mathbf{u}}) &= \mathbf{f}, \\
\nabla\cdot\mathbf{u} &= 0,
\end{aligned} \tag{1}
$$

in the physical domain Ω, with either no-slip Dirichlet boundary conditions or periodic boundary conditions with zero spatial averages. Here $\mathbf{u}$, p, $\mathbf{f}$, denote velocity, pressure and body force, respectively, ν is the kinematic viscosity, χ is a scalar constant known as the relaxation parameter, and $\bar{\mathbf{u}}$ is the solution of the partial differential equation

$$
-\delta^2\Delta\bar{\mathbf{u}} + \bar{\mathbf{u}} = \mathbf{u}, \tag{2}
$$

where $\bar{\mathbf{u}}$ is subject to the same boundary conditions, either Dirichlet or periodic, as were applied to $\mathbf{u}$. In the above filtering equation, δ is a length scale corresponding

Received by the editors July 9, 2018 and, in revised form, September 23, 2020.
2000 *Mathematics Subject Classification.* 65M60, 65M70, 76F65.

to the filter width. Thus, for large δ values, $\bar{\mathbf{u}}$ is smooth and for small δ values, $\bar{\mathbf{u}}$ is close to $\mathbf{u}$. Continuous differential filters were introduced into fluid flow modeling by Germano [14] and used for various fluid flow regularizations, see, for example, [1, 4, 5, 15, 23].

The time relaxation parameter is considered to be a positive quantity, *i.e.*, $\chi > 0$, and has units of inverse time. The term $\chi(\mathbf{u}-\bar{\mathbf{u}})$ aims to drive the unresolved scales to zero exponentially [12]. In working with the TRM, the parameter χ must be specified and scaled appropriately in relation to other parameters in the problem [21].

This paper is organized as follows. In Section 2 we introduce notation and recall some standard results that will be used throughout. Section 3 examines the energy and enstrophy balance of the TRM. Note the energy analysis applies to dimensions $d = 2$ and 3 while enstrophy is considered only for $d = 2$. We show, see equations (11) and (16), that the average rate of dissipation of energy and enstrophy are both increased by a single term proportional to $\chi\delta^2$ compared to the NSE. After characterizing the energy spectrum based on a dimensional analysis of the corresponding inertial ranges for the energy and enstrophy cascades, the results of a fully-resolved spectral-Galerkin simulation are presented to check the dissipation and energy spectrum of the TRM. Note that details of the forcing used in this simulation are given in Appendix A.

Section 4 includes our main results on the practical use of the TRM in finite-element computations. Upon stating the weak formulation, we show that the resulting Oseen problem for the TRM can be approximated numerically using a globally convergent fixed-point iteration. We then compute the Taylor–Green vortex as a benchmark to compare the behavior of the TRM to known NSE dynamics. In particular, the approximate solution obtained by the TRM exhibited the enstrophy increase as desired as well as a near-constant normalized kinetic energy.

The same computational setup, but at a reduced Reynolds number, is then used to numerically solve the sensitivity equations for the TRM using different values of δ and χ. The numerics are found to be consistent with prior sensitivity analysis and illustrate some tradeoffs when tuning these parameters. The paper ends with our conclusions in Section 5 where we provide some additional remarks and also reflect on directions for future study.

2. Preliminaries and Notations

This section presents our functional spaces, preliminary results and notations used in our analysis. While both Dirichlet and periodic boundary conditions will be considered in this paper, we give details here only for the Dirichlet case and note, after accounting for the necessary modifications, that the periodic case is similar. The $L^2(\Omega)$ norm and inner product will be denoted by $\|\cdot\|$ and $(\cdot,\cdot)$. Likewise, the $L^p(\Omega)$ norms and the Sobolev $W_p^k(\Omega)$ norms are denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W_p^k}$, respectively. H^k is used to represent the Sobolev space W_2^k, and $\|\cdot\|_k$ denotes the norm in H^k.

For functions $\mathbf{v}(\mathbf{x},t)$ defined on the entire time interval $(0,T)$, we define

$$\|\mathbf{v}\|_{\infty,k} := \sup_{0<t<T} \|\mathbf{v}(t,\cdot)\|_k \qquad \text{and} \qquad \|\mathbf{v}\|_{m,k} := \left(\int_0^T \|\mathbf{v}(t,\cdot)\|_k^m \, dt \right)^{1/m} .$$