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Shadowing Homoclinic Chains to a Symplectic Critical Manifold

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Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

Abstract. We prove the existence of trajectories shadowing chains of heteroclinic orbits to a symplectic normally hyperbolic critical manifold of a Hamiltonian system. The results are quite different for real and complex eigenvalues. General results are applied to Hamiltonian systems depending on a parameter which slowly changes with rate ε. If the frozen autonomous system has a hyperbolic equilibrium possessing transverse homoclinic orbits, we construct trajectories shadowing homoclinic chains with energy having quasirandom jumps of order ε and changing with average rate of order $\varepsilon|\ln\varepsilon|$. This provides a partial multidimensional extension of the results of A. Neishtadt on the destruction of adiabatic invariants for systems with one degree of freedom and a figure 8 separatrix.

Key Words: Hamiltonian system, homoclinic orbit, shadowing.

AMS Subject Classifications: 37Jxx, 37Dxx, 70Hxx

1 Introduction

Consider a smooth Hamiltonian system (M,ω,H) with phase space M, symplectic form ω and Hamiltonian H. Let $v = J\nabla H$ be the Hamiltonian vector field and ϕ^t the phase flow. Suppose H has a connected symplectic nondegenerate critical manifold N. Then any $z \in N$ is a critical point of H with rank $d^2H(z) = \dim M - \dim N$, and the restriction $\omega|_{T_zN}$ is nondegenerate. We also assume that N is normally hyperbolic, i.e., nonzero eigenvalues of the linearization $\Lambda(z) = Dv(z)$ have nonzero real parts.

Denote by
$$E_z = \{\xi \in T_zM : \omega(\xi,\eta) = 0 \text{ for all } \eta \in T_zN\}$$

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the symplectic complement to T_zN. Since N is symplectic, $T_zM = T_zN \oplus E_z$ and $\omega|_{E_z}$ is nondegenerate. Hence $E_z = E_z^+ \oplus E_z^-$, where $E_z^\pm$ are $\Lambda(z)$-invariant Lagrangian stable and unstable subspaces of E_z corresponding to the eigenvalues with negative and positive real parts respectively.

Let

$$W^\pm(z) = \{x \in M : \lim_{t\to\pm\infty} \phi^t(x) = z\}, \qquad T_zW^\pm(z) = E_z^\pm,$$

be the stable and unstable manifolds of $z \in N$ and

$$W^\pm(N) = \cup_{z\in N} W^\pm(z)$$

the stable and unstable manifolds of N. The intersection $W^+(N) \cap W^-(N) \setminus N$ consists of orbits $\gamma : \mathbb{R} \to M$ homoclinic to N, i.e., heteroclinic from $z_- = \gamma(-\infty)$ to $z_+ = \gamma(+\infty)$. The heteroclinic orbit is called transverse if $T_{\gamma(t)}W^-(z_-) \cap T_{\gamma(t)}W^+(N) = \mathbb{R}\dot\gamma(t)$.

Define a multivalued partially defined symplectic scattering map $\mathcal{F} : N \to N$ by $\mathcal{F}(z_-) = z_+$ if there is a transverse heteroclinic from z_- to z_+. We call a sequence $\sigma = (\sigma_i)_{i\in\mathbb{Z}}$ of transverse heteroclinic orbits a heteroclinic chain if $\sigma_i(+\infty) = \sigma_{i+1}(-\infty) = z_i \in N$. A heteroclinic chain corresponds to an orbit $\mathbf{z} = (z_i)_{i\in\mathbb{Z}}$ of the scattering map. We call the chain strongly nondegenerate if the orbit $\mathbf{z}$ is hyperbolic.

Without loss of generality let $N \subset \Sigma_0 = H^{-1}(0)$. Our goal is to construct, for small μ, orbits $\gamma : \mathbb{R} \to \Sigma_\mu = H^{-1}(\mu)$ shadowing strongly nondegenerate infinite heteroclinic chains. This requires several assumptions which are different for real and complex eigenvalues. For degenerate heteroclinic chains we get weaker results.

Our research is motivated by two classical problems. The first is Poincaré's theory of second species almost collision solutions in celestial mechanics. This application was already discussed in [6,7], so we will be brief. Consider the plane 3 body problem with two small masses of order $\mu \ll 1$. Let the center of mass be at rest and let q_i be the relative positions of small bodies with respect to the large one. Then we obtain the Hamiltonian

$$H_\mu(q,p) = H_0(q,p) + \frac{\mu}{2}|p_1+p_2|^2 - \mu\frac{\alpha_1\alpha_2}{|q_1-q_2|}, \quad q \in (\mathbb{R}^2\setminus\{0\})^2,$$

where

$$H_0 = \sum_{i=1}^2 \left(\frac{|p_i|^2}{2\alpha_i} - \frac{\alpha_i}{|q_i|}\right)$$

is the Hamiltonian of two uncoupled Kepler problems with masses α_i. For $\mu > 0$ there are singularities at double collisions $\Delta = \{q : q_1 = q_2\}$. Fixing an energy level $H_\mu^{-1}(E)$ and performing the Levi-Civita regularization at Δ, we obtain the regularized Hamiltonian $\hat H$ which has a symplectic normally hyperbolic critical manifold $N \subset \hat H^{-1}(0)$ corresponding to Δ. Trajectories of the 3 body problem on $H_\mu^{-1}(E)$ correspond to trajectories of the regularized Hamiltonian on $\Sigma_\mu = \hat H^{-1}(\mu)$. Homoclinic trajectories to N correspond to orbits of the uncoupled Kepler problems with collisions of the small bodies, and trajectories on Σ_μ shadowing heteroclinic chains correspond to almost collision second species