

# Hill-Type Formula and Krein-Type Trace Formula for Hamiltonian Systems

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Dedicated to Prof. Paul H. Rabinowitz with admiration on the occasion of his 80th birthday

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**Abstract.** In this paper, we give a survey on the Hill-type formula and its applications. Moreover, we generalize the Hill-type formula for linear Hamiltonian systems and Sturm-Liouville systems with any self-adjoint boundary conditions, which include the standard Neumann, Dirichlet and periodic boundary conditions. The Hill-type formula connects the infinite determinant of the Hessian of the action functional with the determinant of matrices which depend on the monodromy matrix and boundary conditions. Further, based on the Hill-type formula, we derive the Krein-type trace formula. As applications, we give nontrivial estimations for the eigenvalue problem and the relative Morse index.

**Key Words:** Hill-type formula, trace formula, conditional Fredholm determinant, relative Morse index.

**AMS Subject Classifications:** 34B30, 34L15, 34B09, 37C75, 70H14

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## 1 Introduction

The study of Hill-type formula begins with the original work of Hill [10] in 1877. In his study of the motion of lunar perigee, Hill considered the following equation:

$$\ddot{x}(t) + \theta(t)x(t) = 0, \quad (1.1)$$

where

$$\theta(t) = \sum_{j \in \mathbb{Z}} \theta_j e^{2j\sqrt{-1}t} \quad \text{with } \theta_0 \neq 0$$

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is a real  $\pi$ -periodic function. Let  $\gamma(t)$  be the fundamental solution of the associated first order system of (1.1), that is,

$$\dot{\gamma}(t) = \begin{pmatrix} 0 & -\theta(t) \\ 1 & 0 \end{pmatrix} \gamma(t), \quad \gamma(0) = I_2.$$

Suppose

$$\rho = e^{c\sqrt{-1}\pi}, \quad \rho^{-1} = e^{-c\sqrt{-1}\pi},$$

are the eigenvalues of the monodromy matrix  $\gamma(\pi)$ . In order to compute  $c$ , Hill obtained the following formula which connects the infinite determinant, corresponding to the differential operator, and the characteristic polynomial:

$$\frac{\sin^2(\frac{\pi}{2}c)}{\sin^2(\frac{\pi}{2}\theta_0)} = \det \left[ \left( -\frac{d^2}{dt^2} - \theta_0 \right)^{-1} \left( -\frac{d^2}{dt^2} - \theta \right) \right], \quad (1.2)$$

where the right hand side of (1.2) is the Fredholm determinant. We should point out that the right hand side of the original formula of Hill [10] is a determinant of an infinite matrix. In [10], Hill did not prove the convergence of the infinite determinant, and the convergence was proved by Poincaré [24]. The Hill-type formula for a periodic solution of Lagrangian system on manifold was given by Bolotin [2]. In [3], Bolotin and Treschev studied the Hill-type formula for both continuous and discrete Lagrangian systems with Legendre convexity condition. For the periodic solution of ODE, the Hill-type formula was given by Denk [6]. Please refer more works for Lagrangian systems at [5,7,16,21]. As the beginning of a series of work, Hu and Wang [15] introduced the conditional Fredholm determinant and successfully generalized the Hill-type formula to Hamiltonian systems with  $S$ -periodic boundary conditions. Together with Ou, they obtained the Hill-type formula for Hamiltonian systems with Lagrangian boundary conditions [13]. For the Sturm-Liouville systems, they derived the Hill-type formula with  $S$ -periodic boundary conditions in [12] and with Lagrangian boundary conditions in [17]. By Taylor expansion of the parameterized Hill-type formula, they also derived the Krein-type trace formula [12, 13, 17]. For the non-self-adjoint version of the Hill-type formula, we refer the readers to [16].

The goal of this paper is to derive the Hill-type formula and the Krein-type trace formula for Hamiltonian systems and Sturm-Liouville systems with any self-adjoint boundary conditions, which cover  $S$ -periodic boundary conditions and Lagrangian boundary conditions. The linear Hamiltonian system takes the form

$$\dot{z}(t) = J_n(B(t) + \lambda D(t))z(t), \quad (1.3)$$

where  $B, D \in C([0, T]; \mathcal{S}(2n))$ , and

$$J_n = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$