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Eigenvalues and Eigenfunctions of a Schrödinger Operator Associated with a Finite Combination of Dirac-Delta Functions and CH Peakons*

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Abstract In this paper, we first study the Schrödinger operators with the following weighted function $\sum_{i=1}^{n} p_i\delta(x-a_i)$, which is actually a finite linear combination of Dirac-Delta functions, and then discuss the same operator equipped with the same kind of potential function. With the aid of the boundary conditions, all possible eigenvalues and eigenfunctions of the self-adjoint Schrödinger operator are investigated. Furthermore, as a practical application, the spectrum distribution of such a Dirac-Delta type Schrödinger operator either weighted or potential is well applied to the remarkable integrable Camassa-Holm (CH) equation.

Keywords Schrödinger operator, Boundary conditions, Soliton, Peakon solution, Cammassa-Holm equation.

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1. Introduction

With the recent development of quantum mechanics, nonlinear mathematical physics, and nonlinear waves with peaked soliton (peakon) solutions, some new mathematical problems arise to be solved. The classical Schördinger equation with the Dirac-Delta function $\delta(x)$ or its high-order derivatives $\delta^{(n)}$ as a potential function is the new type of spectral problem endowed with a generalized function or a weighted generalized function. Over eighty years since last centenary, much work has been done on the Schrödinger equation with the $\delta(x)$ or δ'-interaction as a potential function. In [10], the author studied a class of solvable one-dimensional Schrödinger equation in which the Hamiltonian can formally be written as

$$H=-\frac{\mathrm{d}^2}{\mathrm{d}x^2}+\sum_{y\in Y} v_y\delta'(\cdot-y), \tag{1.1}$$

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where Y is a discrete subset of R, finite or infinite, and δ' denotes the derivative of Dirac's δ-function. The existence of model (1.1) was shown by Grossmann et al [11].

In [22], the author mentioned that in the study of the quantized Davey-Stewartson system with two particles ($N = 2$), there was the following Schrödinger equation encountered

$$-\frac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + c\delta'(x)\varphi = E\varphi, \tag{1.2}$$

where c is the coupling constant, and $\delta'(x)$ is the derivative of the Dirac δ-function. The same equation also appeared in [1] and [10]; however, the author of [22] pointed out that the problem was not dealt with correctly, i.e., the boundary conditions for (1.2) in [1] and [10] at the singular point are irrelevant to the $\delta'(x)$-interaction. The appropriate boundary conditions can be replaced by

$$\begin{cases} \varphi(0^+) = \varphi(0^-) = \varphi(0) = 0, \\ \varphi'(0^+) - \varphi'(0^-) = -c\varphi'(0). \end{cases} \tag{1.3}$$

The author in [22] made remarks that the following boundary conditions in [10] and [11]

$$\begin{cases} \varphi'(0^+) = \varphi'(0^-), \\ \varphi(0^+) - \varphi(0^-) = \beta\varphi'(0), \quad \beta \in R, \end{cases} \tag{1.4}$$

are proposed to substitute for (1.3). Afterwards, in [12] the author points out that in [22] an incorrect and extraneous constraint (1.3) is imposed, as the interesting feature of δ' potential precisely implies that φ is not continuous. Because the potential $\delta'(x)$ has been a source of recurring confusion, using the integration by part from $-\varepsilon(x)$ to $+\varepsilon(x)$, the author provides a derivation of the boundary conditions at the singular point $''0''$, this is

$$\begin{cases} -\dfrac{\mathrm{d}^2\varphi}{\mathrm{d}x^2} + c\delta^{(n)}(x)\varphi = E\varphi, \\ \varphi'(0^+) - \varphi'(0^-) = (-1)^n c\bar{\varphi}^{(n)}(0), \\ \varphi(0^+) - \varphi(0^-) = (-1)^{n-1} nc\bar{\varphi}^{(n-1)}(0). \end{cases} \tag{1.5}$$

When $n = 0$, the potential reads $c\delta(x)$, which usually appears in quantum mechanics. The boundary conditions of (1.5) yield the well-known conditions

$$\begin{cases} \varphi(0^+) = \varphi(0^-), \\ \varphi'(0^+) - \varphi'(0^-) = c\varphi(0). \end{cases} \tag{1.6}$$

When $n = 1$, they become

$$\begin{cases} \varphi(0^+) - \varphi(0^-) = c\bar{\varphi}(0), \\ \varphi'(0^+) - \varphi'(0^-) = c\bar{\varphi}'(0). \end{cases} \tag{1.7}$$

We notice that these conditions are different from the boundary conditions in [22] and the boundary conditions in [1, 10].