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# Bogdanov-Takens Bifurcation in a Host-parasitoid Model

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**Abstract** In this paper, we study a host-parasitoid model with Holling II Functional response, where we focus on a special case: the carrying capacity $K_2$ for parasitoids is equal to a critical value $\frac{r_1}{\eta}$. It is shown that the model can undergo Bogdanov-Takens bifurcation. The approximate expressions for saddle-node, Homoclinic and Hopf bifurcation curves are calculated. Numerical simulations, including bifurcation diagrams and corresponding phase portraits, are also given to illustrate the theoretical results.

**Keywords** Host-parasitoid model, Holling II functional response, Bogdanov-Takens bifurcation.

**MSC(2010)** 34C23, 34C25.

## 1. Introduction

Following the pioneering work of Fisher [2], many mathematical models have been proposed to describe and stop or reverse biological invasions ( [5,7,8,11]). Predators are able to stop or reverse invasions, as shown first by Owen and Lewis [9], and a slowdown of invasions can be obtained if the functional response is linear (Type I).

In order to study the invasion and biological control of leaf-mining microlepidopteron, which attacks horse chestnut trees in Europe. Magal *et al.* [6] developed a host-parasitoid model with Holling II Functional response as follows

$$\begin{aligned}\dot{u} &= r_1 u(1-\frac{u}{K_1}) - \frac{\eta uv}{1+\eta hu},\\ \dot{v} &= r_2 v(1-\frac{v}{K_2}) + \frac{\gamma\eta uv}{1+\eta hu},\end{aligned} \tag{1.1}$$

where $u(t)$ and $v(t)$ denote densities of the hosts (leafminers *microlepidopteron*) and generalist parasitoids (*Minotetrastichus frontalis*) at time $t$, respectively. $r_1$ and $r_2$ represent the intrinsic growth rate of the hosts and parasitoids, respectively, $K_1$ and $K_2$ represent the carrying capacity of the hosts population and parasitoids population, respectively. $\eta$ is the encounter rate of hosts and parasitoids, $\gamma$ is the conversion rate of parasitoids, $h$ describes the harvesting time. $r_i$, $K_i(i=1,2)$, $\gamma$, $\eta$, $h$ are all positive constants. Magal *et al.* [6] analyzed the number and stability of equilibria in system (1.1), and showed some complex dynamical behaviors, such as

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the existence of a cusp, an unstable limit cycle and a homoclinic loop, by numerical simulations. However, the complex nonlinear dynamics and bifurcation phenomena still remain unknown, which is the subject of this paper.

For simplicity, we first nondimensionalize system (1.1) with the following scaling

$$\bar{t} = r_1 t, \quad \bar{x} = \frac{u}{K_1}, \quad \bar{y} = \frac{r_2 v}{r_1 K_2},$$

dropping the bars, model (1.1) becomes

$$\begin{aligned} \dot{u} = x &= x(1 - x - \frac{by}{a+x}), \\ \dot{v} &= y(\delta - y + \frac{cx}{a+x}), \end{aligned} \tag{1.2}$$

where

$$a = \frac{1}{K_1 \eta h}, \quad b = \frac{K_2}{K_1 r_2 h}, c = \frac{\gamma}{r_1 h}, \quad \delta = \frac{r_2}{r_1},$$

and $a, b, c, \delta$ are all positive. In this paper, we focus on a special case of system (1.2): $\delta = \frac{a}{b}$, i.e., the carrying capacity $K_2$ for parasitoids is equal to a critical value $\frac{r_1}{\eta}$. It is shown that the model can undergo Bogdanov-Takens bifurcation. The approximate expressions about saddle-node, Homoclinic and Hopf bifurcation curves are calculated. Numerical simulations, including bifurcation diagrams and corresponding phase portraits, are also given to illustrate the results.

This paper is organized as follows. In section 2, we analyse the existence and types of equilibria in model (1.2) when $\delta = \frac{a}{b}$. In section 3, we show the existence of Bogdanov-Takens bifurcation, and some numerical simulations are also given to illustrate the theoretical results. The paper ends with a brief discussion.

## 2. Equilibria and their types

By the biological implication, we only consider system (1.2) in $\mathbb{R}_+^2 = \{(x,y)|x \geqslant 0, y \geqslant 0\}$. It is easy to see that the positive invariant and bounded region of system (1.2) is

$$\Omega = \{(x,y)|0 \leq x \leq 1, 0 \leq y \leq \delta + \frac{c}{a+1}\}.$$

It is easy to see that system (1.2) always has three boundary equilibria $(0,0)$, $(1,0)$ and $(0,\delta)$ for all permissible parameters. The Jacobian matrix of system (1.2) at any equilibrium $E(x,y)$ of system (1.2) takes the following form

$$J(E) = \begin{pmatrix} 1 - 2x - \frac{aby}{(a+x)^2} & -\frac{bx}{a+x} \\ \frac{acy}{(a+x)^2} & \frac{a}{b} - 2y + \frac{cx}{a+x} \end{pmatrix},$$

and

$$\mathrm{Det}(J(E)) = (1 - 2x - \frac{aby}{(a+x)^2})(\frac{a}{b} - 2y + \frac{cx}{a+x}) + \frac{abcxy}{(a+x)^3},$$

$$\mathrm{Tr}(J(E)) = 1 + \frac{a}{b} - 2(x+y) + \frac{cx}{a+x} - \frac{aby}{(a+x)^2}.$$

It implies that $E(x,y)$ is an elementary equilibrium if $\mathrm{Det}(J(E)) \neq 0$, a hyperbolic saddle if $\mathrm{Det}(J(E)) < 0$, or a degenerate equilibrium if $\mathrm{Det}(J(E)) = 0$, respectively.

We can easily get the following results about the types of the boundary equilibria.