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Oscillation of Second Order Impulsive Differential Equations with Nonpositive Neutral Coefficients*

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Abstract In this work, sufficient conditions are established for a class of nonlinear second order neutral impulsive differential equations to have oscillatory solutions with nonpositive neutral coefficient. Our results extend and complement some of the known results in the literature. Examples are given to illustrate our results.

Keywords Oscillation, Nonoscillation, Neutral differential equation, Impulse, Nonlinear.

MSC(2010) 34K, 34K40, 34K45, 34K11.

1. Introduction

Consider the class of second order impulsive nonlinear neutral differential equations of the form:

$$(E)\begin{cases} [x(t)+p(t)x(t-\tau)]''+g(t,x(t),x(t-\sigma))=0,\ t\neq\theta_k,\ t\geq t_0, & (1.1)\\ x(\theta_k^+)=I_k(x(\theta_k)),\ k\in\mathbb{N}, & (1.2)\\ x'(\theta_k^+)=J_k(x'(\theta_k)),\ k\in\mathbb{N}, & (1.3)\end{cases}$$

where $\tau,\sigma\in\mathbb{N}$, $0\leq t_0<\theta_1<\cdots<\theta_k<\cdots$ with $\lim_{k\to\infty}\theta_k=\infty$ and $\theta_{k+1}-\theta_k>\rho=\max\{\tau,\sigma\}$. Throughout our work, we assume that the following hypotheses hold:

(A_1) $g\in C([t_0-\rho,\infty)\times\mathbb{R}\times\mathbb{R},\mathbb{R})$, $ug(t,u,v)>0$ for $uv>0$, $\frac{g(t,u,v)}{h(v)}\geq q(t)$ for $v\neq 0$, where $q(t)\in C([t_0-\rho,\infty),\mathbb{R}_+)$ and $q(t)\not\equiv 0$ on all interval of the form $(\theta_k,\theta_{k+1}]$, $k\geq 1$, $xh(x)>0$ for all $x\neq 0$ and $h'(x)\geq\varepsilon>0$;

(A_2) $I_k, J_k\in C(\mathbb{R},\mathbb{R})$, $I_k(0)=0=J_k(0)$ and there exist positive numbers c_k, c_k^*, d_k, d_k^*, such that $c_k^*\leq\frac{I_k(u)}{u}\leq c_k$, $d_k^*\leq\frac{J_k(u)}{u}\leq d_k$, $k\in\mathbb{N}$;

(A_3) $p\in PC(\mathbb{R}_+,\mathbb{R})$ and $p(t)$, $p'(t)$ are left continuous on $(\theta_k,\theta_{k+1}]$, $k\geq 1$ such that $p(\theta_k^+)=d_kp(\theta_k)$, $p'(\theta_k^+)=d_kp'(\theta_k)$.

In the literature (see for e.g. [11]), the impulse operators are often treated as **under control**, that is, one may expect that either the impulse act as a control and

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cease the oscillation of the system, or operate to keep the system oscillating. In particular, impulse can make oscillating systems become nonoscillating and conversely by the imposition of suitable impulse control (see for e.g. [5]- [9], [13]- [17], [23], [27]).

One of the important application of second order differential equations with impulse is in impact theory. Billiard-type systems, models describing viscoelastic bodies colliding, systems with delay and impulse are more appropriate to apply (see for e.g. [10]). Of course, some extra conditions are required while we study impulsive equations (see for e.g. [2, 3, 21, 22, 26, 28]) to that of nonimpulsive equations. Furthermore, it is more challenging to study nonlinear neutral equations as we find a class of second order delay differential equations as special cases. In this respect, by using comparison technique, the second order impulsive neutral differential equations

$$(E^*)\begin{cases}[r(t)(v(t)+p(t)v(t-\tau))']'+q(t)v(t-\sigma)=0,\ t\neq\theta_k,\ t\geq t_0,\\ v(\theta_k^+)=(1+d_k)v(\theta_k),\ k\in\mathbb{N},\\ v'(\theta_k^+)=(1+d_k)v'(\theta_k),\ k\in\mathbb{N},\end{cases}$$

has been studied by Li et al. [13], where $\tau,\sigma\in\mathbb{N}$, $q(t)>0$, $r(t)>0$, $b_k>-1$ and $p(t)=p\geq 0$; they have extended and generalised the work of [6] to impulse equations.

By using the Riccati transfomation technique, Bonotto et al. [4] have considered the second order neutral differential equations with impulse of the form:

$$(E^*)\begin{cases}[r(t)(v(t)+p(t)v(t-\tau))']'+f(t,v(t),v(t-\sigma))=0,\ t\neq\theta_k,\ t\geq t_0,\\ v(\theta_k)=I_k(v(\theta_k^-)),\ k\in\mathbb{N},\\ v'(\theta_k)=J_k(v'(\theta_k^-)),\ k\in\mathbb{N},\end{cases}$$

where $\tau,\sigma\in\mathbb{N}$, $p\in PC([t_0,\infty),\mathbb{R}_+)$, $r(t)>0$, $\theta_{k+1}-\theta_k>\sigma=\max\{\tau,\sigma\}$ and $c_k^*\leq\frac{I_k(u)}{u}\leq c_k$, $J_k(u)=d_ku$, $k\in\mathbb{N}$, c_k^*, c_k, $d_k>0$ and $f(t,v(t),v(t-\sigma))\geq q(t)f(x(t-\sigma))$, and $f(x)=x$. In this work, the authors have extended and generalised the work of [12] to impulsive equations in the range $0\leq p(t)<1$.

However, it seems that there is no known results regarding the oscillation of second order impulsive neutral differential equations when the neutral coefficient $p(t)\leq 0$. More exactly, the existing literature does not provide any criteria which ensure oscillation of all solutions of (E) when $p(t)\leq 0$. In view of this motivation, our aim in this paper is to present sufficient conditions which ensure that all solutions of (E) are oscillatory.

Definition 1.1. A real valued continuous function $x(t)$ is said to be a solution of (E) satisfying the initial condition, if the following conditions are satisfied

1. $x(t)=\psi(t)$ for $t_0-\rho\leq t\leq t_0$, $x(t)\in C^2[t_0,\infty,\mathbb{R})$ and $t\neq\theta_k,k\in\mathbb{N}$;
2. $y(t)=x(t)+p(t)x(t-\tau)\in C^1([t_0,\infty),\mathbb{R})$ and $y'(t)\in C^1([t_0,\infty),\mathbb{R})$, $t\neq\theta_k,t\neq\theta_k+\tau,t\neq\theta_k+\sigma,k\in\mathbb{N}$ and satisfies (1.1);
3. $x(\theta_k^+),x(\theta_k^-),x'(\theta_k^+)$ and $x'(\theta_k^-)$ exist, $x(\theta_k^-)=x(\theta_k)$, $x'(\theta_k^-)=x'(\theta_k)$ and satisfies (1.2) and (1.3) respectively.

Definition 1.2. A nontrivial solution $x(t)$ of (E) is said to be nonoscillatory, if there exists a point $t_0\geq 0$ such that $x(t)$ has a constant sign for $t\geq t_0$. Otherwise, the solution $x(t)$ is said to be oscillatory. (E) is oscillatory, if all its solutions are oscillatory.