Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 2, Number 4, December 2020, 565–571 DOI:10.12150/jnma.2020.565

Qualitative Analysis and Periodic Cusp Waves to a Class of Generalized Short Pulse Equations*

Zhenshu Wen[1,†] and Keqin Zhao[1]

Abstract In this paper, we qualitatively study periodic cusp waves to a class of generalized short pulse equations, which are of the general form of three special generalized short pulse equations, from the perspective of dynamical systems. We show the existence of smooth periodic waves, periodic cusp wave and compactons, obtain exact expression of periodic cusp wave and illustrate the limiting process of periodic cusp wave from smooth periodic waves.

Keywords Generalized short pulse equations, Periodic cusp waves, Periodic waves, Compactons, Bifurcation.

MSC(2010) 34A05, 35C07, 36C09, 35A24.

1. Introduction

In 2018, N. W. Hone, Novikov and Wang [4] obtained three generalized short pulse equations

$$u_{xt} = u + 2uu_{xx} + 2u_x^2, \tag{1.1}$$
$$u_{xt} = u + 2uu_{xx} + u_x^2, \tag{1.2}$$
$$u_{xt} = u + 4uu_{xx} + u_x^2, \tag{1.3}$$

which possess an infinite hierarchy of local higher symmetries, when they were classifying nonlinear partial differential equations of second order of the general form

$$u_{xt} = u + c_0u^2 + c_1uu_x + c_2uu_{xx} + c_3u_x^2 + d_0u^3 + d_1u^2u_x + d_2u^2u_{xx} + d_3uu_x^2. \tag{1.4}$$

Note that if we take $c_0 = c_1 = c_2 = c_3 = 0$, $d_0 = d_1 = 0$, $d_2 = 1$, and $d_3 = 2$, Eq.(1.4) becomes the following short pulse equation

$$u_{xt} = u + \frac{1}{3}(u^3)_{xx}, \tag{1.5}$$

which was derived by Schäfer and Wayne [10] as a model of ultra-short optical pulses in nonlinear media. In [9], the authors showed that the short pulse equation (1.5)

†the corresponding author.
Email address: wenzhenshu@hqu.edu.cn(Z. Wen)
[1]School of Mathematical Sciences, Huaqiao University, Quanzhou, Fujian 362021, China
*The authors were supported by by the National Natural Science Foundation of China (No. 11701191), and the Fundamental Research Funds for the Central Universities (No. ZQN-802).

is integrable, in the sense that it admits a Lax pair and a recursion operator that generates infinitely many commuting symmetries, and they also found a hodograph-type transformation which connects Eq.(1.5) with the sine-Gordon equation.

As suggested in [4], short pulses and their properties are a subject of current interest in nonlinear optics and electrodynamics, both theoretically and experimentally. For instance, a rigorous justification of the short pulse equation, starting from a quasilinear Klein-Gordon equation (a toy model for Maxwells equations) was given in [8]. Moreover, for electrons accelerated in short laser pulses, it was shown recently that, due to quantum effects, the radiation reaction can be quenched by suitably tuning the pulse length, although the lengths required are currently out of experimental reach [3].

In this paper, based on the forms of Eqs. (1.1), (1.2) and (1.3), we focus on a class of generalized short pulse equations, which have the following general form

$$u_{xt} = u + \alpha u u_{xx} + \beta u_x^2, \tag{1.6}$$

where we assume the parameters $\alpha > 0$ and $\beta > 0$, for conveniently. Obviously, taking $\alpha = 2, \beta = 2$, Eq.(1.6) becomes Eq.(1.1). Similarly, taking $\alpha = 2, \beta = 1$, Eq.(1.6) becomes Eq.(1.2), and taking $\alpha = 4, \beta = 1$, Eq.(1.6) becomes Eq.(1.3). We intend to study the solutions to the general form (1.6) qualitatively from the perspective of dynamical systems [2,6,7,11–14,17–24] and consequently, the results about three special forms (1.1), (1.2) and (1.3) follow immediately.

2. Phase portrait

Employing the traveling wave transformation $u(x,t) = \varphi(\xi)$, $\xi = x - ct$, where $c > 0$ is the wave speed, we can convert Eq.(1.6) into the following ordinary differential equation

$$(\alpha\varphi + c)\varphi'' + \varphi + \beta(\varphi')^2 = 0. \tag{2.1}$$

Introducing $y = \varphi'$, we obtain a planar system

$$\begin{cases} \frac{d\varphi}{d\xi} = y, \\ \frac{dy}{d\xi} = \frac{-\varphi - \beta y^2}{\alpha\varphi + c}, \end{cases} \tag{2.2}$$

with first integral

$$\begin{aligned} H(\varphi, y) =& (\alpha\varphi + c)^{\frac{2\beta}{\alpha}} y^2 + \frac{2}{\alpha(\alpha + 2\beta)}(\alpha\varphi + c)^{\frac{2\beta}{\alpha}+1} - \frac{c}{\alpha\beta}(\alpha\varphi + c)^{\frac{2\beta}{\alpha}}, \\ & \text{for } \varphi \geq -\frac{c}{\alpha}, \\ H(\varphi, y) =& (-\alpha\varphi - c)^{\frac{2\beta}{\alpha}} y^2 - \frac{2}{\alpha(\alpha + 2\beta)}(-\alpha\varphi - c)^{\frac{2\beta}{\alpha}+1} - \frac{c}{\alpha\beta}(-\alpha\varphi - c)^{\frac{2\beta}{\alpha}}, \\ & \text{for } \varphi < -\frac{c}{\alpha}. \end{aligned} \tag{2.3}$$

Transformed by $d\xi = (\alpha\varphi + c)d\tau$, system (2.2) becomes a Hamiltonian system

$$\begin{cases} \frac{d\varphi}{d\tau} = (\alpha\varphi + c)\ y, \\ \frac{dy}{d\tau} = -\varphi - \beta y^2. \end{cases} \tag{2.4}$$