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Global Regularity of the Logarithmically Supercritical MHD System in Two-dimensional Space

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Abstract In this paper, we study the global regularity of logarithmically supercritical MHD equations in 2 dimensional, in which the dissipation terms are $-\mu\Lambda^{2\alpha}u$ and $-\nu\mathcal{L}^{2\beta}b$. We show that global regular solutions in the cases $0<\alpha<\frac{1}{2}, \beta>1, 3\alpha+2\beta>3$.

Keywords Logarithmically supercritical, MHD system, Global regularity.

MSC(2010) 35Q35, 35B65, 76D05.

1. Introduction

We consider the two-dimensional logarithmically supercritical magnetohydrodynamics (MHD) system:

$$u_t + u\cdot\nabla u + \nabla\pi + \mu\Lambda^{2\alpha}u - b\cdot\nabla b = 0, \tag{1.1}$$

$$b_t + u\cdot\nabla b + \nu\mathcal{L}^{2\beta}b - b\cdot\nabla u = 0, \tag{1.2}$$

$$(u,b)(x,0) = (u_0,b_0) \ in\ \mathbb{R}^2, \tag{1.3}$$

$$\mathrm{div}u = \mathrm{div}b = 0. \tag{1.4}$$

where $u=u(x,t)\in\mathbb{R}^2$ is the unknown velocity field, $b=b(x,t)\in\mathbb{R}^2$ is the magnetic field, and $\pi=\pi(x,t)\in\mathbb{R}$ represents the pressure. $\alpha,\beta\geq 0$ are real parameters. $\Lambda=(-\Delta)^{1/2}$ is defined in terms of the Fourier transform $\widehat{\Lambda f}(\xi)=|\xi|\widehat{f}(\xi)$, and $\mathcal{L}^{2\beta}$ defined through a Fourier transform,

$$\widehat{\mathcal{L}^{2\beta}f}(\xi) = m(\xi)\hat{f}(\xi), m(\xi) = \frac{|\xi|^{2\beta}}{g^2(|\xi|)}, \beta\in\mathbb{R}^+.$$

with $g:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ a radially symmetric, non-decreasing function such that $g\geq 1$.

When

$$\mathcal{L}^{2\beta} = \Lambda^{2\beta}.$$

For the system (1.1)-(1.4), We identify the case $\mu=\nu=0$ as the GMHD system with zero velocity and zero magnetic diffusion respectively (so called ideal MHD equations). The author in [1] studied the global existence of a weak solution when $\alpha\geq\frac{1}{2}+\frac{n}{4}$, $\alpha+\beta\geq 1+\frac{n}{2}, n\in\mathbb{R}^3$. In [2], the author showed that the GMHD

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equations exists a unique global smooth solution when $\alpha, \beta \geq \frac{1}{2} + \frac{n}{4}$, There are some results [3–8] about the existence of the strong solution.

We want to improve the lower bound on the power of the fractional Laplacian in the dissipative term of the generalized Navier-Stokes equations seems extremely difficult, the author introduced the notion of "logarithmic supercriticality" in [9,10], and also proved the global regularity of the solution. the author improved that the results [2] by using the notion of "logarithmic supercriticality" in [11], it were improved that the solution is globally regular in [12,13].

Tran, Yu and Zhai [14] proved that the solutions are globally regular in the following conditions:

$$(1)\alpha \geq \frac{1}{2}, \beta \geq 1; \quad (2) 0 \leq \alpha \leq \frac{1}{2}, 2\alpha + \beta > 2; \quad (3)\alpha \geq 2, \beta = 0.$$

it were improved that the solution is globally regular of the GMHD equations in [15–19], and there are some results [20–22] about logarithmic type.

Now we focus on our study. The authors in [16] got a global regular solution under the assumption that $0 \leq \alpha < \frac{1}{2}, \beta \geq 1, 3\alpha + 2\beta > 3$. In this paper, the dissipation term $-\nu\Lambda^{2\beta}b$ has been replaced by general negative-definite operator $-\nu\mathcal{L}^{2\beta}b$ by using the definition in [23], and in the proof, we will use the condition in [24] on g such that there exists an absolute constant $c \geq 0$ satisfying

$$g^2(\tau) \leq c \ln(e + \tau).$$

Theorem 1.1. *Let $0 < \alpha < \frac{1}{2}, \beta > 1, 3\alpha + 2\beta > 3$, Suppose $u_0, b_0 \in H^s$ with $s \geq 2$ and $divu_0 = divb_0 = 0$ in $\mathbb{R}^2$. Then the problem (1.1)-(1.4) exists the solution (u, b) satisfying*

$$u, b \in L^\infty(0, T; H^s), \ u \in L^2(0, T; H^{s+\alpha}), \ b \in L^2(0, T; H^{s+\beta'}). \tag{1.5}$$

for any $T > 0$ and $\beta > \beta' > 1$.

Remark 1.1. When $\alpha + \beta > 2, s > 2$, the author in [14] prove the global regularity.

2. Preliminaries

In this section, we will review some known facts and elementary inequalities that will be used frequently later.

Lemma 2.1. ***(ϵ-Young inequality)*** *If a and b are nonnegative real numbers and p and q are real numbers greater than 1 such that $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$ab \leq \frac{\epsilon a^p}{p} + \epsilon^{-\frac{q}{p}} \frac{b^q}{q},$$

the equality holds if and only if $a^p = b^q$.

Lemma 2.2. ***(Gagliardo-Nirenberg inequality [25, 26])*** *Let u belong to L^q and its derivatives of order m, $\Lambda^m u$, belong to L^r, $1 \leq q, r \leq \infty$. For the derivatives $\Lambda^j u$, $0 \leq j < m$, the following inequalities hold*

$$\|\Lambda^j u\|_{L^p} \leq C \|\Lambda^m u\|_{L^r}^{\alpha} \|u\|_{L^q}^{1-\alpha}, \tag{2.1}$$