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Qualitative Analysis of Crossing Limit Cycles in Discontinuous Liénard-Type Differential Systems*

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Abstract In this paper, we investigate qualitative properties of crossing limit cycles for a class of discontinuous nonlinear Liénard-type differential systems with two zones separated by a straight line. Firstly, by applying left and right Poincaré mappings we provide two criteria on the existence, uniqueness and stability of a crossing limit cycle. Secondly, by geometric analysis we estimate the position of the unique limit cycle. Several lemmas are given to obtain an explicit upper bound for the amplitude of the limit cycle. Finally, a predator-prey model with nonmonotonic functional response is studied, and Matlab simulations are presented to show the agreement between theoretical results and numerical analysis.

Keywords Discontinuous Liénard-type differential system, crossing limit cycle, existence, uniqueness, stability, position.

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1. Introduction

In the real world, many problems from mechanics, physics and engineering are discontinuous in nature, such as forcing terms of electro-magnetic field and controls in engineering. The mathematical modeling can be described by differential equations with discontinuous right-hand sides, called as discontinuous differential systems. Up to now, there have been rich achievements on the basic properties of solutions and the stability theory (see monograph [6] for example). The main topics have been become the analysis of existence and uniqueness of periodic orbits, number and bifurcation of limit cycles in qualitative theory of planar systems of ordinary differential equations. An important type of planar systems is of the form

$$\frac{d^2x}{dt^2} + f(x)\frac{dx}{dt} + g(x) = 0, \tag{1.1}$$

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called as Liénard differential equation which exhibits notably rich dynamics. The equivalent system of (1.1)

$$\frac{dx}{dt} = y - F(x), \qquad \frac{dy}{dt} = -g(x) \tag{1.2}$$

with $F(x) = \int_0^x f(s)ds$, and the more general form

$$\frac{dx}{dt} = h(y) - F(x), \qquad \frac{dy}{dt} = -g(x) \tag{1.3}$$

are both called as Liénard-type systems. Many planar models in physical applications, chemical reaction and biological models can be transformed into the Liénard forms (1.2) or (1.3). The problem on limit cycles of (1.2) and (1.3), such as the (non)existence, uniqueness, number and relative position have been widely studied, see [1,4,10,11,15-16,18-22] for example. The existence is often proved by the well known Poincaré-Bendixson theorem by constructing a trapping zone where a limit cycle is located. The following conditions appear frequently in the literatures of the study of limit cycles of (1.2) and (1.3):

- (i) $xg(x) > 0$ for $x \neq 0$, and $G(x) = \int_0^x g(s)ds$ with $G(\pm\infty) = +\infty$;
- (ii) there exist $x_1 < 0 < x_0$ such that $F(x_1) = F(0) = F(x_0) = 0$, $xF(x) > 0$ for $x \in (x_0, +\infty) \cup (-\infty, x_1)$ and $xF(x) < 0$ for $x \in (x_1, x_0)$;
- (iii) $F'(x) > 0$ for $x \in (x_0, +\infty) \cup (-\infty, x_1)$, and $F(\pm\infty) = \pm\infty$;
- (iv) $h'(y) > 0$ for $y \in \mathbb{R}$, and $h(\pm\infty) = \pm\infty$.

Note that the theory of smooth systems cannot be directly applied to the discontinuous case. In fact, there may be appearing sliding solutions, grazing solutions or impact solutions. Hence it is necessary to study the discontinuous differential systems in theory and applications. About the investigation of existence, uniqueness, number and bifurcation of limit cycles, one can see [2-3,5-6,7-9,12-14] for example. In this paper, we consider a Liénard-type system (1.3) allowing for discontinuity, where the discontinuities occur along a straight line (called as a discontinuity line). The analysis shows that the system has no sliding solutions. Therefore, we focus on the existence, uniqueness and stability of a crossing limit cycle. Here the crossing limit cycle is defined as a limit cycle with trajectory intersecting the discontinuity line at finite points, and the time parameter representation of this kind of limit cycles is usually continuous but possesses discontinuous derivative on the discontinuity line.

Another classical topic in the qualitative theory of planar systems of ordinary differential equations is to estimate the position and amplitude of limit cycles. However, there are few papers involving the upper bound of amplitude of limit cycles for Liénard-type systems [1,3-4,15-16,18]. Recently, Yang and Zeng [18] studied an upper bound of the amplitude of a unique limit cycle for (1.2) with symmetry under some conditions as follows

- (i) $f(x)$ has a unique positive zero $a_1 > 0$ and $f(x)(x - a_1) > 0$ for $x > 0, x \neq a_1$;
- (ii) $f(x)/g(x)$ is monotone increasing on $(a_1, +\infty)$;
- (iii) $\int_0^x F(s)g(s)ds > 0$ for sufficiently large $x > 0$.