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Steady-state Solution for Reaction-diffusion Models with Mixed Boundary Conditions*

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Abstract In this paper, we deal with a diffusive predator-prey model with mixed boundary conditions, in which the prey population can escape from the boundary of the domain while predator population can only live in this area and can not leave. We first investigate the asymptotic behaviour of positive solutions and obtain a necessary condition ensuring the existence of positive steady state solutions. Next, we investigate the existence of positive steady state solutions by using maximum principle, the fixed point index theory, L_p-estimation, and embedding theorems, Finally, local stability and uniqueness are obtained by linear stability theory and perturbation theory of linear operators.

Keywords Mixed boundaries, local stability, uniqueness.

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1. Introduction

A number of biologists and mathematicians have devoted themselves to ecological mathematical models and have achieved many impressive results [1, 3, 5–7, 9, 12, 17, 18, 20, 22–25, 27, 28] since Lotka [11] and Volterra [26] established the following classical predator model

$$\begin{cases} \dfrac{\partial u}{\partial t} = r_1 u \left(1 - \dfrac{u}{k_1}\right) - cuv, \\ \dfrac{\partial v}{\partial t} = r_2 v \left(1 - \dfrac{v}{k_2}\right) + mcuv, \end{cases} \tag{1.1}$$

where u and v represent the densities of prey and predator, respectively, r_1, r_2 are the intrinsic growth rates of the prey and predator, respectively, $r_1 u(1 - \frac{u}{k_1})$ represents prey's natural growth rate, k_1 represents the maximum number of prey that the environment can support, $r_2 v(1 - \frac{v}{k_2})$ denotes predator's natural growth rate, k_2 is the maximum number of predator that the environment can support. Moreover, cu represents the number of prey that can be captured by the unit predator per

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unit time, which is also called the functional response function, and m is predator's transmission rate after capturing prey. This model has some obvious deficiencies and has been improved by using some suitable functional response function $f(u,v)$ instead of the simple function cu in different applications.

In this paper, we shall investigate a diffusive Lotka-Volterra model under the Neumann boundary condition combined with the third type of boundary condition: the prey species satisfy the Neumann boundary condition, while the predator species satisfy the third type of boundary condition. Let Ω be a bounded domain in $\mathbb{R}^n$ $(n \geqslant 1)$ with smooth boundary $\partial\Omega$, and ω be the outward unit normal vector on $\partial\Omega$. Define $\frac{mv}{\gamma+u^2}$ as the response function of the prey species, and $\frac{cu}{\gamma+u^2}$ as the response function of the predator species after predation. Let d_1 and d_2 be the diffusion coefficients of the prey and predator, respectively, which implies that when the population is unevenly distributed in the region, the species spontaneously return to a uniform state. Denote by a positive constant α the proportion of predators escaping from the regional boundary $\partial\Omega$. Thus, we shall investigate the following diffusive prey-predator model

$$\begin{cases} \dfrac{\partial u}{\partial t} - d_1\Delta u = au - u^2 - \dfrac{muv}{\gamma+u^2}, & (x,t)\in\Omega\times(0,\infty), \\ \dfrac{\partial v}{\partial t} - d_2\Delta v = bv - v^2 + \dfrac{cuv}{\gamma+u^2}, & (x,t)\in\Omega\times(0,\infty), \\ \dfrac{\partial u}{\partial \omega} = \dfrac{\partial v}{\partial \omega} + \alpha v = 0, & (x,t)\in\partial\Omega\times(0,\infty), \\ u(x,0) = u_0(x), & x\in\Omega, \\ v(x,0) = v_0(x), & x\in\Omega, \end{cases} \tag{1.2}$$

where $u_0(x)$, $v_0(x) : \Omega \to \mathbb{R}^n$ are continuous initial functions. The steady state problem of (1.2) is

$$\begin{cases} -d_1\Delta u = au - u^2 - \dfrac{muv}{\gamma+u^2}, & x\in\Omega, \\ -d_2\Delta v = bv - v^2 + \dfrac{cuv}{\gamma+u^2}, & x\in\Omega, \\ \dfrac{\partial u}{\partial \omega} = \dfrac{\partial v}{\partial \omega} + \alpha v = 0, & x\in\partial\Omega. \end{cases} \tag{1.3}$$

One of our purposes is to investigate the existence of positive steady state solutions of system (1.2), which is equivalent to the existence of positive solutions of system (1.3). We first notice that (1.3) has a trivial solution $\mathbf{0} = (0,0)$ and a semi-trivial solutions $u_* = (a,0)$. We shall employ the in-cone fixed point index theory to calculate indexes at points $\mathbf{0}$ and u_*, and then make use of the maximum principle, L_p-estimation and embedding theorem to show that system (1.3) has at least one positive solution.

The paper is organized as follows: In Section 2, we give the necessary conditions ensuring the existence of positive steady state solutions and the asymptotic behaviour of the positive solution. In Section 3, we investigate the asymptotic behaviours of positive solutions of (1.2) and give some necessary conditions ensuring the existence of positive steady-state solutions of (1.2). Section 4 is devoted to the existence of positive steady state solutions of system (1.2). Section 5 is devoted to the local stability and uniqueness of the positive solution of system (1.3).