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Global Structure of Planar Quadratic Semi-Quasi-Homogeneous Polynomial Systems*

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Abstract This paper study the planar quadratic semi-quasi-homogeneous polynomial systems(short for PQSQHPS). By using the nilpotent singular points theorem, blow-up technique, Poincaré index formula, and Poincaré compaction method, the global phase portraits of such systems in canonical forms are discussed. Furthermore, we show that all the global phase portraits of PQSQHPS can be-classed into six topological equivalence classes.

Keywords Semi-quasi-homogeneous, quadratic system, singular point, global phase portraits.

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1. Introduction

In this paper, we consider the planar polynomial differential systems of the form

$$\dot{x} = P(x,y), \ \dot{y} = Q(x,y), \tag{1.1}$$

where P and Q are polynomials, $\dot{x}, \dot{y}$ are the first derivatives with regard to the time variable t. We call (1.1) polynomial differential system of degree n , if n is the maximum degree of P and Q.

The planar polynomial differential system (1.1) is called to be semi-quasi homogeneous, if there exist $s_1, s_2, d_1, d_2 \in \mathbb{N}^+$ and $d_1 \neq d_2$ such that for any arbitrary $\lambda \in \mathbb{R}^+$,

$$P(\lambda^{s_1}x, \lambda^{s_2}y) = \lambda^{s_1-1+d_1}P(x,y), \ Q(\lambda^{s_1}x, \lambda^{s_2}y) = \lambda^{s_2-1+d_2}Q(x,y). \tag{1.2}$$

We call $w = (s_1, s_2, d_1, d_2)$ the weight vector of system(1.1), s_1 and s_2 the weight exponents of system (1.1), and d_1, d_2 the weight degree with respect to weight exponents s_1 and s_2. In particular, system (1.1) is a semi-homogeneous system when $s_1 = s_2$, and, (1.1) is a quasi-homogeneous system when $d_1 = d_2$.

The weight vector $w = (s_1, s_2, d_1, d_2)$ of planar semi-quasi-homogeneous system (1.1) is not unique (see [19]) and a weight vector $w_m = (s_1^*, s_2^*, {d_1}^*, {d_2}^*)$ is called to be a minimal weight vector of (1.1) if any other weight vector $w = (s_1, s_2, d_1, d_2)$ of (1.1) satisfies $s_1^* \le s_1, \ s_2^* \le s_2, \ d_1^* \le d_1, \ d_2^* \le d_2$.

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Obviously, the homogeneous and semi-homogeneous systems are respectively the special types of the quasi-homogeneous and semi-quasi homogeneous systems, which have attracted lots of interests due to their special properties. In fact, a large number of scholars have explored these two kinds of systems from different aspects, such as the centers [6, 7, 12], limit cycles [5], first integral [3], canonical forms [8], phase portraits of the systems [4, 15] and so on. The research on quasi-homogeneous systems has also become a hot topic since the 21st century. In the study of planar quasi-homogeneous polynomial systems, the identification of center from monodromy singularity, the number and location of limit cycles and various analytical first integrals (including polynomial first integral, rational first integral and Liouville first integral) are discussed. For example, the integrability of the planar quasi-homogeneous system was studied in the literature [1, 2, 10, 11], the bifurcation of limit cycles in quasi-homogeneous centers was discussed in [13, 14], the algorithm for obtaining the canonical forms of all given quasi-homogeneous but non-homogeneous systems was provided in [11]. Later, some authors used this algorithm to obtain the canonical form of quasi-homogeneous system of degree 2–6, and made further efforts to analyse the dynamic behavior of them. For instance, the authors of [17] analyzed the quintic quasi-homogeneous system, and, the global phase portraits and the first integral properties of such systems were obtained by further analysis after the canonical forms were given.

The semi-quasi-homogeneous polynomial differential systems are the generalizations of semi-homogeneous and quasi-homogeneous systems, and appear in many fields of the natural sciences. For example, in Newton's laws of motion when the mass of a particle is potion dependent, we have the motion equation $\ddot{x}-a\dot{x}^2-bx^3=0$ which can be changed to the semi-quasi-homogeneous polynomial differential systems $\dot{x}=y, \dot{y}=ay^2+bx^3$, with the weight vector $w_m=(2,3,2,4)$, see [19] and the references therein.

However, as far as we know, there are a few of results on the semi-quasi-homogeneous polynomial differential systems. Because they do not have such special properties as the semi-homogeneous and quasi-homogeneous systems that it is more difficult to be studied. Here we list two important results about the semi-quasi-homogeneous systems. The authors of [18] gave a criterion for the non-existence of a rational first integral of a semi-quasi-homogeneous system by using Kowalevsky exponent. Zhao studied the limit cycles of semi-quasi homogeneous systems in [21]. He gave some sufficient conditions for the nonexistence and existence of periodic orbits, and, gave a lower bound for the maximum number of limit cycles of such systems.

Very resent, inspired by [11], the authors of [19] established an algorithm to obtain all the semi-quasi homogeneous systems with a given degree and got all the canonical forms of semi-quasi homogeneous systems of degree 2 and 3. After obtaining the canonical forms of the planar semi-quasi-homogeneous systems, we naturally pay more attention to the global dynamic behavior of them. Therefore, in this paper, we will study the global structure of planar quadratic semi-quasi-homogeneous polynomial systems (short for PQSQHPS) on the basis of [19] and give the global phase portrait structures of PQSQHPS in the sense of topological equivalence.