Journal of Nonlinear Modeling and Analysis http://jnma.ca; http://jnma.ijournal.cn
Volume 1, Number 3, September 2019, 301–305 DOI:10.12150/jnma.2019.301

New Proofs of Monotonicity of Period Function for Cubic Elliptic Hamiltonian*

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Abstract In [1] S.-N. Chow and J. A. Sanders proved that the period function is monotone for elliptic Hamiltonian of degree 3. In this paper we significantly simplify their proof, and give a new way to prove this fact, which may be used in other problems.

Keywords Periodic function, elliptic Hamiltonian, Abelian integrals.

MSC(2010) 34C07, 34C08, 37G15.

1. Introduction

Consider the cubic elliptic Hamiltonian function $H(x,y) = \frac{y^2}{2} + P_3(x)$, there P_3 is a polynomial of degree 3, the corresponding quadratic Hamiltonian system is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = y, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -P_3'(x).$$

Suppose that the origin is a non-degenerate center, so we can write $P_3(x) = \frac{1}{2}x^2 - \frac{a}{3}x^3$, where $a \neq 0$. If we write the closed orbit, surrounding the origin, by

$$\gamma_h \subset H^{-1}(h) = \{(x,y)|H(x,y) = h\},$$

then, from the first equation of the system, we can write the period function by

$$T(h) = \oint_{\gamma_h} \frac{1}{y}\,\mathrm{d}x, \tag{1.1}$$

where $y = y(x,h)$ is defined by $H(x,y) = h$. Note that by the scaling $(x,y) \mapsto (\frac{x}{a}, \frac{y}{a})$, the period function does not change, hence without loss of generality we can suppose that γ_h is defined by

$$H(x,y) = \frac{y^2}{2} + A(x) = h, \quad A(x) = \frac{x^2}{2} - \frac{x^3}{3}, \tag{1.2}$$

and the corresponding Hamiltonian system is

$$\begin{aligned} \tfrac{\mathrm{d}x}{\mathrm{d}t} &= y, \\ \tfrac{\mathrm{d}y}{\mathrm{d}t} &= -x + x^2 = x(x-1). \end{aligned} \tag{1.3}$$

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[*] The second author was supported by National Natural Science Foundation of China (11771282).

The continuous family of ovals is $\{\gamma_h \subset H^{-1}(h), 0 < h < \frac{1}{6}\}$, γ_h shrinks to the center at $(x, y) = (0, 0)$ when $h \to 0^+$, and γ_h expand to the homoclinic loop Γ related to the saddle at $(x, y) = (1, 0)$ when $h \to \frac{1}{6}^-$.

Theorem 1.1 (Theorem 3.8 of [1]). *The period function $T(h)$ is monotone for $0 < h < \frac{1}{6}$.*

For more information about the study of period functions, see Section 2.4 of [2], for example.

2. A simple proof of Theorem 1.1

We first give a very simple proof of Theorem 1.1 by using Picard-Fuchs equation.

Let

$$I_k(h) = \oint_{\gamma_h} x^k y \, \mathrm{d}x, \quad k = 0, 1, 2, \cdots, \tag{2.1}$$

then by using $yy_h = 1$ and (2.1) we have

$$I_k'(h) = \oint_{\gamma_h} \frac{x^k}{y} \, \mathrm{d}x, \quad k = 0, 1, 2, \cdots. \tag{2.2}$$

Lemma 2.1. *The following equalities hold:*

$$\begin{aligned} 5I_0 &= 6hI_0' - I_1', \\ 7I_1 &= I_0 + (6h-1)I_1', \end{aligned} \tag{2.3}$$

where $I_k = I_k(h), I_k' = I_k'(h)$.

Proof. From (2.1), (1.2) and (2.2) we have

$$I_k = \oint_{\gamma_h} \frac{x^k y^2}{y} \, \mathrm{d}x = \oint_{\gamma_h} \frac{x^k(2h - x^2 + \frac{2}{3}x^3)}{y} \, \mathrm{d}x = 2hI_k' - I_{k+2}' + \frac{2}{3} I_{k+3}'.$$

On the other hand, by using integration by parts and the fact that $\mathrm{d}y = \frac{x^2 - x}{y} \, \mathrm{d}x$ we have

$$I_k = \oint_{\gamma_h} x^k y \mathrm{d}x = -\frac{1}{k+1} \oint_{\gamma_h} x^{k+1} \mathrm{d}y = \frac{1}{k+1} \oint_{\gamma_h} \frac{x^{k+1}(x - x^2)}{y} \, \mathrm{d}x = \frac{I_{k+2}' - I_{k+3}'}{k+1}. \tag{2.4}$$

Eliminating I_{k+3}' from the above two equalities, we obtain

$$(2k+5)I_k = 6hI_k' - I_{k+2}'.$$

Taking $k = 0, 1$, we find

$$\begin{aligned} 5I_0 &= 6hI_0' - I_2', \\ 7I_1 &= 6hI_1' - I_3'. \end{aligned} \tag{2.5}$$

By integrating $(x - x^2)y \, \mathrm{d}x = y^2 \, \mathrm{d}y$ along γ_h we get $I_1(h) \equiv I_2(h)$, hence the first equation of (2.5) gives the first equality of (2.3). Taking $k = 0$ in (2.4) we have