Journal of Nonlinear Modeling and Analysis
Volume 1, Number 3, September 2019, 307–318
http://jnma.ca; http://jnma.ijournal.cn
DOI:10.12150/jnma.2019.307

Traveling Wave Solutions of a Fourth-order Generalized Dispersive and Dissipative Equation*

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Abstract In this paper, we consider a generalized nonlinear forth-order dispersive-dissipative equation with a nonlocal strong generic delay kernel, which describes wave propagation in generalized nonlinear dispersive, dissipation and quadratic diffusion media. By using geometric singular perturbation theory and Fredholm alternative theory, we get a locally invariant manifold and use fast-slow system to construct the desire heteroclinic orbit. Furthermore we construct a traveling wave solution for the nonlinear equation. Some known results in the literature are generalized.

Keywords Dispersive-dissipative equation, geometric singular perturbation, traveling waves, heteroclinic orbit.

MSC(2010) 35Q53, 74J30, 34D25.

1. Introduction

In this paper, we are concerned with the existence of traveling wave solution for the generalized fourth-order dispersive and dissipative equation

$$\frac{\partial u}{\partial t}+\alpha u^n(f*u)\frac{\partial u}{\partial x}+\beta\frac{\partial^2 u}{\partial x^2}+\gamma\frac{\partial^3 u}{\partial x^3}+s(\frac{\partial}{\partial x}(u\frac{\partial u}{\partial x}))+\delta\frac{\partial^4 u}{\partial x^4}=0, \tag{1.1}$$

where $n\geq 1$, α, β, γ, s and δ are constant coefficients. u is a function of space x and time t, α is the nonlinear convective coefficient, β is the diffusion coefficient, γ is the dispersion coefficient, s is the backward quadratic diffusion coefficient and δ is the stable coefficient. Here, partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial t}$ indicate the corresponding partial differentiation with respect to spatial variable x and time variable t, respectively. $\frac{\partial^3 u}{\partial x^3}$, $\frac{\partial}{\partial x}(u\frac{\partial u}{\partial x})$ and $\frac{\partial^4 u}{\partial x^4}$ represent to dispersion effect term, backward quadratic diffusion term and the stable term, respectively. We take $f*u$ to be the following spatial-temporal convolution

$$(f*u)(x,t)=\int_{-\infty}^{t}\int_{-\infty}^{+\infty}f(x-y,t-s)u(y,s)dyds,$$

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*Supported by the Natural Science Foundation of China (Grant Nos. 11771185, 11871251 and 11801231).

here the function f satisfies the normalization conditions

$$f(t) \geq 0 \ for \ t \geq 0, \text{and} \int_{-\infty}^{t} \int_{+\infty}^{-\infty} f(x,t)dxdt = 1,$$

such that the kernel f doesn't affect the spatial-temporal uniform steady states.

The Eq.(1.1) describes wave propagation in generalized nonlinear dispersive, dissipation and quadratic diffusion media. It can be discovered in the context of Benard-Marangoni convection in shallow layers, thin liquid films, and so on [7]. Eq.(1.1) has many applications, for example, is governing evolution equation for the propagation of weak nonlinear waves in fluid-filled thick viscoelastic tubes for arterial blood flow. We point out that, if the parameters are chosen as different values, some famous equations can be derived from Eq.(1.1). For instance, if $n = 0$, $f * u = u$, $\beta = s = \delta = 0$, Eq.(1.1) becomes the Korteweg-de Vries (KdV, for short) equation [8]. As is known to that the KdV equation has been widely studied due to its significance in stratified internal wave, physical contexts, plasma physics and its applications in weakly nonlinear dispersive physical system [2,5,9,15,16]. When $n = 0$, $f * u = u$, $s = \varepsilon = 0$, Eq.(1.1) becomes the Burgers-KdV equation, which was first proposed the standard form by Feudel and Steudel [4] when they proved that the equation has no prolongation structure.

Mansour [11] considered a fourth order Burgers-KdV equation and proved the existence of traveling wave solutions. By using the dynamical systems theory, especially based upon the geometric singular perturbation theory and invariant manifold theorem, Mansour [12] constructed the traveling wave solutions of a nonlinear dispersive-dissipative equation. In many cases, differential equations with time delay can reflect the real natural phenomena. The delay is an significant factor that can not be ignored, which can make the steady state of the system change. Shang and Du [14] discussed the existence of traveling wave solutions to a nonlinear dispersive and dissipative equation

$$\frac{\partial u}{\partial t} + \alpha u^n (f * u) \frac{\partial u}{\partial x} + \beta \frac{\partial^2 u}{\partial x^2} + \gamma \frac{\partial^3 u}{\partial x^3} + s \frac{\partial}{\partial x}(u \frac{\partial u}{\partial x}) = 0, \tag{1.2}$$

where $n \geq 1$, α, β, γ, and s are constant coefficients.

If the time delay disappears in Eq.(1.1), i.e., $f * u = u$, Mansour [12] found the existence of homolinic orbit of Eq.(1.1) by applying the method of the Melnikov function. We will get the existence of the heteroclinic orbit by using the invariant manifold on the phase plane. In the later part of the article, we also discuss the Eq.(1.1) with spatial-temporal delay, which describes the state that the system variables depend on the system at a certain time or in a certain historical period. If we choose $\delta = 0$, Eq.(1.1) becomes the Eq.(1.2) discussed by Shang and Du [14]. Our results agree well with the corresponding ones in [14]. In the case that Eq.(1.1) without delay, Shang and Du [14] obtained the existence of the heteroclnic orbit by constructing the triangular invariant set. However, our approach overcomes the difficulties that Eq.(1.1) adds the fourth order term by constructing the three pyramid invariant to get the heteroclinic orbit.

The remaining part of this article is organized as following. In section 2, we will construct the existence of traveling wave solutions of Eq.(1.1) without delay. In section 3, we will investigate Eq.(1.1) with a nonlocal delay. Using geometric singular perturbation theory [3,6] and Fredholm theorem, we get a locally invariant manifold and seek the heteroclinic orbit in this slow manifold. Furthermore we construct a traveling wave solution of Eq.(1.1). In section 4, we give a conclusion.