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Dynamics of a Diffusive SIR Epidemic Model with Time Delay*

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Abstract This paper is devoted to a reaction-diffusion system for a SIR epidemic model with time delay and incidence rate. Firstly, the nonnegativity and boundedness of solutions determined by nonnegative initial values are obtained. Secondly, the existence and local stability of the disease-free equilibrium as well as the endemic equilibrium are investigated by analyzing the characteristic equations. Finally, the global asymptotical stability are obtained via Lyapunov functionals.

Keywords Diffusion, SIR epidemic model, time delay, basic reproduction number, stability, Lyapunov functional.

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1. Introduction

In this paper, we consider the following SIR epidemic model:

$$\begin{cases} S_t(x,t) - d_S\Delta S(x,t) = am - aS(x,t) - S(x,t)f(I(x,t-\tau)), & x \in \Omega, \\ I_t(x,t) - d_I\Delta I(x,t) = S(x,t)f(I(x,t-\tau)) - (a+c)I, & x \in \Omega, \\ R_t(x,t) - d_R\Delta R(x,t) = cI(x,t) - aR(x,t), & x \in \Omega, \\ \dfrac{\partial S}{\partial \mathbf{n}} = \dfrac{\partial I}{\partial \mathbf{n}} = \dfrac{\partial R}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

for $t \geq 0$, where d_S, d_I, d_R, a, c, m and τ are positive constants, the density functions $S(x,t)$, $I(x,t)$ and $R(x,t)$ represent the numbers of susceptible, infective and recovered individuals at position x and time t, respectively, and the parameters d_S, d_I, and d_R are their diffusion coefficients. The constant am is the recruitment rate of the susceptible population, a is a natural death rate for all the susceptible, infective and recovered population, c is the recovery rate of the infective individuals, and τ is the latent period of the disease. The constant m can be interpreted as a

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carrying capacity, or maximum possible population size. Δ denotes the Laplacian operator on $\mathbb{R}^N$, $\mathbf{n}$ is the outward unit normal vector on $\partial\Omega$. The homogeneous Neumann boundary condition means that the two species have zero flux across the boundary $\partial\Omega$. In practical use, there are various types of the incidence term $Sf(I)$. The common types include bilinear incidence (or mass action incidence) bSI (see, for example, [4,6,7,16,18,22,27]), standard incidence bSI/m (see, for example, [11]), and saturated incidence $bSI/(1+\alpha I)$ (see, for example, [5,11,12,17,19,20,24,26,28]), where b and α are positive constants. Throughout this paper, we always assume that the function $f(\cdot)$ is strictly monotone increasing, positive, and continuously differentiable on $[0,\infty)$ and satisfies the conditions

$$f(0)=0, \quad f(x)\le f'(0)x$$

for all $x>0$, and

$$\left[\frac{f(x)}{x}-\frac{f(y)}{y}\right][f(x)-f(y)]\le 0 \tag{1.2}$$

for all x, $y>0$. The initial conditions of system (1.1) are given as

$$\begin{cases} S(x,0)=S_0(x), \quad R(x,0)=R_0(x), \\ I(x,\theta)=I_0(x,\theta) \quad \text{for all} \quad \theta\in[-\tau,0]. \end{cases} \tag{1.3}$$

For a SIR epidemic model without diffusion (i.e., $d_S=d_I=d_R=0$), Wang [13] studied the existence, uniqueness and some estimates of a global solution, and also investigated the long time behavior of solutions to an initial-boundary value problem in a half space. Similarly, in this paper we can define a number R_0 (so-called the basic reproduction number) such that the disease-free equilibrium is stable when $R_0<1$. But for the case $R_0>1$ the endemic equilibrium is asymptotically stable. Kumar, Narayan and Reddy [14] studied the local asymptotical stability of the disease-free equilibrium and endemic equilibrium, and obtained the existence of the Hopf bifurcation at the positive equilibrium, Greenhalgh [8] studied the some SEIBS epidemiological models with vaccination and temporary immunity are considered. First of all, previously published work is reviewed. A general model with a constant contact rate and a density dependent death rate is examined. The model is reformulated in terms of the proportions of susceptible, incubating, infectious, and immune individuals. The equilibrium and stability properties of this model are examined, assuming that the average duration of immunity exceeds the infectious period. There is a threshold parameter R_0, and the disease can persist if and only if R_0, exceeds one. The disease-free equilibrium always exists and is locally stable if $R_0<1$ and unstable if $R_0>1$. Conditions are derived for the global stability of the disease-free equilibrium. For $R_0>1$, the endemic equilibrium is unique and locally asymptotically stable.

This paper is organized as follows. In section 2 we consider the nonnegativity and boundedness and show that all solutions of system (1.1) is nonnegative and bounded for all $t\ge 0$; Sections 3 is devoted to the local stability of equilibria of model (1.1). The global stability of the endemic equilibrium when $R_0<1$ and $R_0>1$ is proved in section 4, Numerical simulations are provided in section 5. In the paper, we denote by $\mathbb{N}$ (respectively, $\mathbb{R}_+$) the set of all the positive integers (respectively, nonnegative real numbers), and $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. Denote by $H^k(\Omega,\mathbb{R}_+)$ $(k\ge 0)$ the Sobolev space of the nonnegative L^2-functions $f(x)$ defined on Ω whose derivatives