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Sufficient Conditions of Blow-up and Bound Estimations of Blow-up Time for a Parabolic Equation in Multi-dimensional Space*

Xiaona Miao[1], Xiaomin Xue[1] and Fushan Li[1,†]

Abstract In this paper, we establish some sufficient conditions on the heat source function and the heat conduction function of the parabolic equation to guarantee that $u(\boldsymbol{x}, t)$ blows up at finite time, and give upper and lower bounds of the blow-up time in multi-dimensional space.

Keywords Sufficient conditions, blow-up, upper and lower bounds, multi-dimensional space.

MSC(2010) 35K55, 35K60.

1. Introduction

In this paper, we deal with the initial-boundary value problem

$$\begin{cases} u_t - \Delta u = f(u), & (\boldsymbol{x}, t) \in \Omega \times (0, t^*), \\ u = 0, & (\boldsymbol{x}, t) \in \Gamma_0 \times (0, t^*), \\ \dfrac{\partial u}{\partial \boldsymbol{n}} = g(u), & (\boldsymbol{x}, t) \in \Gamma_1 \times (0, t^*), \\ u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}) \geq 0, & \boldsymbol{x} \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^N$ $(N \geq 2)$ with smooth boundary $\Gamma := \partial\Omega$, $\Gamma = \Gamma_0 \cup \Gamma_1$, meas $(\Gamma_0 \cap \Gamma_1) = 0$, meas $(\Gamma_0) \geq 0$, meas $(\Gamma_1) > 0$ and $\boldsymbol{n} = (n_1, n_2, \cdots, n_N)$ is the unit outward normal vector on Γ_1, $u_0 \in C^1\left(\overline{\Omega}\right)$, $u_0(\boldsymbol{x}) \geq 0$, $u_0 \not\equiv 0$, and t^* is the blow-up time if blow-up occurs. From the physical standpoint, f is the heat source function and g is the heat conduction function transmitting into interior of Ω from the boundary Γ_1.

The blow-up phenomena of solutions to evolution partial differential equations has received considerable attentions in recent years. For the work in this area, the reader can refer to the book Quittner [9] and papers [1, 4]. Many methods have been used to determine the blow-up of solutions and to indicate an upper bound of the blow-up time. To our knowledge, the first work on lower bound of t^* was given by Weissler [10, 11]. Recently, a number of papers deriving lower bound of t^* in various problems have appeared (see [2, 3, 6–8, 12, 13] and the references therein).

†the corresponding author.
Email address:fushan99@163.com (F. Li), xnmiao0422@163.com (X. Miao), xuexiaomin31415926@163.com (X. Xue)
[1]School of Mathematical Sciences, Qufu Normal University, Qufu, Shangdong 273165, China
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The blow-up for nonlinear equations with *Neumann boundary* conditions has received considerable attentions. Payne and Schaefer [6] considered homogeneous equation without heat source term

$$u_t = \Delta u, \quad (\boldsymbol{x}, t) \in \Omega \times (0, t^*).$$

Under suitable nonlinear conditions, they deduced a lower bound of the blow-up time when blow-up occurs only in *three-dimensional* space. Mizoguchi [5] studied the semilinear heat equation with a power function heat source term

$$u_t = \Delta u + u^p, \quad (\boldsymbol{x}, t) \in \Omega \times (0, T),$$

and showed that if u blows up at $t = T$, then $|u(t)|_\infty \leq C(T-t)^{-\frac{1}{p-1}}$ for some $C > 0$. Payne etc [8] considered heat equation with general heat source term

$$u_t = \Delta u - f(u) \quad \boldsymbol{x} \in \Omega, t \in (0, t^*),$$

and established conditions on nonlinearities to guarantee that $u(\boldsymbol{x}, t)$ blows up at some finite time t^*. Moreover, an upper bound for t^* was derived. Under some more restrictive conditions, a lower bound for t^* was derived only in *three-dimensional* space. Li and Li [2] investigated nonhomogeneous divergence form parabolic equation

$$u_t = \sum_{i=1}^{N} (a_{ij}(\boldsymbol{x}) u_{x_i})_{x_j} - f(u), \quad t \in (0, t^*), \ \boldsymbol{x} = (x_1, x_2, \cdots, x_N) \in \Omega,$$

and gave the conditions on nonlinearities to guarantee that $u(\boldsymbol{x}, t)$ exists globally or blows up at some finite time respectively. If blow-up occurs, they obtained upper and lower bounds of the blow-up time, but the lower bound of t^* was valid only in *three-dimensional* space.

Motivated by the above work, we intend to study the blow-up phenomena for problem (1.1). It is well known that the data f and g may greatly affect the behavior of $u(\boldsymbol{x}, t)$ with the development of time. The larger the heat source function f and conduction function g are, the greater possibility the blow-up will occur, and the earlier blow-up time will be. The main contributions of this paper are: (i) the conditions of blow-up are derived naturally by means of calculation process, and some examples satisfying the conditions are given; (ii) the lower bound of blow-up time is given under the conditions that ensure occurrence of blow-up phenomena; (iii) the lower bound of blow-up time is obtained in multi-dimensional space which improves the situation discussed in three-dimensional space.

The present work is organized as follows. In Section 2, we derive the conditions on f, g to ensure that the solutions blow up at finite time and obtain an upper bound of the blow-up time. In Section 3, under the conditions on f and g that guarantee the occurrence of blow-up, we get a lower bound of blow-up time t^* in multi-dimensional space.

2. Blow-up and upper bound estimation of t^*

In order to derive the sufficient conditions for blow-up phenomena and the upper bound of blow-up time, we first give the following calculation. From the physical