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On the Integrability and Equivalence of the Abel Equation and Some Polynomial Equations*

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Abstract In this paper, first of all we give the necessary and sufficient conditions of the center of a class of planar quintic differential systems by using reflecting function method, and provide a simple proof of this results. Secondly, We use the reflecting integral to research the equivalence of the Abel equation and some complicated equations and derive their center conditions and discuss their integrability.

Keywords Center condition, integrability, reflecting integral, equivalence.

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1. Introduction

In this paper, we will consider the Abel equations of the form

$$\frac{dr}{d\theta} = A(\theta)r^2 + B(\theta)r^3, \tag{1.1}$$

where $A(\theta)$, $B(\theta)$ are continuous functions. The main reason why we are interested in this Abel equations is that they are closely related to planar vector fields. There are many classes of planar systems which are in some sense equivalent to some Abel equations [1–4, 6, 7, 14, 15]. The first class is planar polynomial systems of the form $x' = -y + p$, $y' = x + q$ with homogeneous polynomials p and q of degree k. The second class is the Liénard systems : $x' = y$, $y' = -f(x)y - g(x)$, they can be transformed to the Abel (1.1) [15]. The third class is the system

$$\begin{cases} x' = -y + x(P_n(x,y) + P_{2n}(x,y)), \\ y' = x + y(P_n(x,y) + P_{2n}(x,y)), \end{cases} \tag{1.2}$$

where $P_k(x,y) = \sum_{i+j=k} p_{ij}x^i y^j$, $p_{i,j}$ $(i,j = 0,1,2,...,k,\ k = n, 2n)$ are real constants.

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In polar coordinates, the system (1.2) becomes

$$\frac{d\rho}{d\theta} = (P_n(\cos\theta, \sin\theta) + P_{2n}(\cos\theta, \sin\theta)\rho^n)\rho^{n+1}. \tag{1.3}$$

Taking $r = \rho^n$, then (1.3) becomes (1.1) with $A(\theta) = nP_n$, $B(\theta) = nP_{2n}$. The origin is a center for the two-dimensional system (1.2) if and only if all solutions of the Abel equation (1.1) starting near the origin are periodic with period 2π , *i.e.*, all the solutions nearby are closed: $r(2\pi) = r(0)$. In this case, we say that $r = 0$ is a center of the Abel equation.

The Abel equations have been investigated over the years. In the papers [1–4,6,7,14,15] and others, the authors Alwash and Lloyd presented some center conditions for the Abel equations and give the composition conditions [2,3] under which the Able equation has a center. Yomdin [6] and Yang [15] give an asymptotic expansion of the solutions of Abel equations and some center conditions.

In this paper, in the first section, we use reflecting function method [9,17,18] to derive the center conditions for a class of planar quintic differential systems and provide a simple proof of this results. In the second section, we give the integrability conditions of some polynomial differential equations by using its reflecting integrals [17], and establish the equivalence between the polynomial equations and some complicated equations and judge when do these complicated equations have a center at the origin.

Now, I briefly introduce the concept of the reflecting function and reflecting integral which will be used throughout the rest of this article.

Consider differential system

$$x' = X(t, x), \ (t \in I \subset R, \, x \in D \subset R^n, \, 0 \in I) \tag{1.4}$$

which has a continuously differentiable right-hand side and general solution $\varphi(t; t_0, x_0)$.

Definition 1.1. [9] For system (1.4), $F(t, x) := \varphi(-t, t, x)$ is called its **Reflecting function**.

By this, for any solution $x(t)$ of (1.4), we have $F(t, x(t)) = x(-t), F(0, x) = x$ and $F(t, x)$ is a reflecting function of system (1.4), if and only if, it is a solution of the Cauchy problem

$$F_t + F_x X(t, x) + X(-t, F) = 0, F(0, x) = x. \tag{1.5}$$

By [9,18], if system (1.4) is 2ω-periodic with respect to t, and $F(t, x)$ is its reflecting function, then $T(x) := F(-\omega, x)$ is the Poincaré mapping of (1.4) over the period $[-\omega, \omega]$, and the solution $x = \varphi(t; -\omega, x_0)$ of (1.4) defined on $[-\omega, \omega]$ is 2ω-periodic if and only if x_0 is a fixed point of $T(x)$. Thus, we can use the method of reflecting function to study the existence and stability of the periodic solutions of the differential systems (1.4) [5,9–13,17,18].

Definition 1.2. [9] If the reflecting functions of two differential systems coincide in their common domain, then these systems are said to be **Equivalent**.

Definition 1.3. [17] If $\Delta(t, x)$ is a unequal identically to zero solution of the partial differential system

$$\Delta_t(t, x) + \Delta_x(t, x)X(t, x) - X_x(t, x)\Delta(t, x) = 0, \tag{1.6}$$

then $\Delta(t, x)$ is called a **Reflecting integral** of (1.4).