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Weak Solutions of a Reaction Diffusion System with Superdiffusion and Its Optimal Control*

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Abstract The existence and uniqueness of weak solutions to the 2-dimensional reaction diffusion system with superdiffusion and the optimal control of such model are investigated in this paper. Fractional function spaces, Galerkin approximation method and Gronwall inequality are used to obtain the existence and uniqueness of weak solutions. On this basis, an optimal control problem of such superdiffusive system is further considered by using the minimal sequence.

Keywords Weak solutions, optimal control, reaction-diffusion, Riesz operator.

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1. Introduction

From a microscopic point of view, diffusion is usually described as the random motion of the individual particles among some media. Normal diffusion is often referred to as the Gaussian process, also known as the Brownian motion (Wiener process). In such process, the waiting time distribution and the jump length distribution between successive two jumps of particles must have finite moments. Due to the center limit theorem, it is characterized by the mean square displacement of a typical particle growing linearly with time $< x^2(t) >\propto t$. Meanwhile, the anomalous diffusion are frequently observed, for example, in Refs. [6, 7, 10, 16–18] and references therein, in which the mean square displacement violates the linear relation with time and universally obeys the power-law relation, scaled as $< x^2(t) >\propto t^\gamma$. When $0 < \gamma < 1$, it is called the subdiffusion, with the waiting time distribution having infinite moments, in this case the particle will wait for long times before next jumping, such phenomena could be found in porous media, polymers and gels, etc. When $\gamma = 1$, it just corresponds to the normal diffusion. If $1 < \gamma < 2$, it is said to be the superdiffusion, which is featured as the limiting result of Lévy flight, with the jumping length distribution having infinite moments. On such occasion, the particle will execute very long jumps. Such process could occur in the precesses of plasmas, turbulence, surface diffusion and motion of animals, etc.

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Since the anomalous diffusion has been theoretically speculated and experimentally found in nature, models, such as continuous time random walk, transport on fractals and fractional Brownian motion, have been established to manifest the anomalous processes, and differential equations with fractional order derivatives are presented to serve as a more appropriate mathematical tool to describe anomalous diffusion and the transmission dynamics of complex systems [2]. As opposed to the normal diffusion governed by the reaction-diffusion system with the standard Laplace operator Δ in it, the superdiffusion process is described by the diffusive system with the Laplace operator replaced by the fractional order operator ∇^α $(\alpha = \frac{2}{\gamma})$. So it is reasonable and necessary to take the behaviors of superdiffusive systems into consideration.

In recent years, many researchers consider the following reaction diffusion system with superdiffusion

$$\begin{cases} \frac{\partial u}{\partial t} - k_1 \nabla^\alpha u = f(u,v), & Q_T := (0,T] \times \Omega, \\ \frac{\partial v}{\partial t} - k_2 \nabla^\alpha v = g(u,v), & Q_T := (0,T] \times \Omega, \\ u(x,y,0) = u_0, & v(x,y,0) = v_0, \\ u(x,y,t) = 0, v(x,y,t) = 0, \Sigma_T := [0,T] \times \mathbb{R}^2 \backslash \Omega, \end{cases} \tag{1.1}$$

where k_1 and k_2 are diffusion coefficients, $1 < \alpha \le 2$ and Ω is a bounded open domain in $\mathbb{R}^2$. Reaction terms f and g can be expressed by distinct coupling reactions between u and v, which satisfy the Lipschitz conditions, i.e.,

$$|f(u_1,v_1) - f(u_2,v_2)| \le L||(u_1,v_1) - (u_2,v_2)||, \tag{1.2}$$

$$|g(u_1,v_1) - g(u_2,v_2)| \le L||(u_1,v_1) - (u_2,v_2)||, \tag{1.3}$$

for $\forall (u_1,v_1),(u_2,v_2) \in \mathbb{R}^2$, here L is a Lipschitz constant, with $f(0,0) = 0,\ g(0,0) = 0$. The fractional operator ∇^α is a sequential Riesz fractional order operator in space [15], and could be given in [24] as follows:

$$\nabla^\alpha u = \frac{\partial^\alpha u}{\partial |x|^\alpha} + \frac{\partial^\alpha u}{\partial |y|^\alpha} = -\frac{1}{2\cos(\pi\alpha/2)} \left[({}_xD_L^\alpha u + {}_xD_R^\alpha u) + ({}_yD_L^\alpha u + {}_yD_R^\alpha u) \right].$$

Here $\frac{\partial^\alpha u}{\partial |x|^\alpha} = -\frac{1}{2\cos(\pi\alpha/2)} ({}_xD_L^\alpha u + {}_xD_R^\alpha u)$, ${}_xD_L^\alpha$ and ${}_xD_R^\alpha$ are defined, respectively

$$\begin{aligned} {}_xD_L^\alpha u =& \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_a^x (x-s)^{1-\alpha} u(s,y,t) ds, \\ {}_xD_R^\alpha u =& \frac{1}{\Gamma(2-\alpha)} \frac{\partial^2}{\partial x^2} \int_x^b (s-x)^{1-\alpha} u(s,y,t) ds, \end{aligned}$$

where $\alpha \in (1,2)$, $\Gamma(\cdot)$ is the Gamma function. The right Riemann-Liouville fractional derivative $\frac{\partial^\alpha u}{\partial |y|^\alpha}$ can be defined similarly.

The existence and uniqueness of a solution are the basic theory of differential equations. However, research on the existence and uniqueness of the variational solution to the evolution equations with superdiffusion was delayed as a result of difficulties caused by fractional operators. For example, the fractional operator is non-local and the adjoint of a fractional differential operator is not the negative of itself. But, Ervin and Roop [3,4,19,20] introduced fractional derivative spaces and