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On Non-commuting Sets in a Finite p-group with Derived Subgroup of Prime Order

WANG YU-LEI[1] AND LIU HE-GUO[2]

(*1. Department of Mathematics, Henan University of Technology, Zhengzhou, 450001*)

(*2. Department of Mathematics, Hubei University, Wuhan, 430062*)

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Abstract: Let G be a finite group. A nonempty subset X of G is said to be non-commuting if $xy \neq yx$ for any $x, y \in X$ with $x \neq y$. If $|X| \geq |Y|$ for any other non-commuting set Y in G, then X is said to be a maximal non-commuting set. In this paper, we determine upper and lower bounds on the cardinality of a maximal non-commuting set in a finite p-group with derived subgroup of prime order.

Key words: finite p-group, non-commuting set, cardinality

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In this paper, p is always a prime. The notation used is standard, see [1].

Let G be a finite group. A nonempty subset X of G is said to be non-commuting if $xy \neq yx$ for any $x, y \in X$ with $x \neq y$. And further, if $|X| \geq |Y|$ for any other non-commuting set Y in G, then X is said to be a maximal non-commuting set in G. We denote by $nc(G)$ the cardinality of a maximal non-commuting set in G, and denote by $cc(G)$ the minimal number of abelian subgroups covering G. Then we can obtain

$$nc(G) \leq cc(G) \leq (nc(G)!)^2.$$

Further, Pyber[2] has shown that there exists some constant c such that

$$cc(G) \leq |G : \zeta G| \leq c^{nc(G)}.$$

Mason[3] has shown that any finite group G can be covered by at most $[|G|/2]+1$ abelian subgroups, thus

$$nc(G) \leq [|G|/2] + 1.$$

For an extraspecial p-group G with order p^{2n+1}, in the case of $p = 2$, Isaacs has obtained that $nc(G) = 2n + 1$ (see [4]). In the case of p being odd, Chin[5] has determined the

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E-mail address: yulwang@163.com (Wang Y L).

following bounds for $nc(G)$,

$$np + 1 \le nc(G) \le (p(p-1)^n - 2)/(p-2).$$

Now we determine upper and lower bounds for the cardinality of a maximal non-commuting sets in a finite p-groups with derived subgroup of prime order.

Lemma 1[6] *Let G be a finite p-group with derived subgroup of order p. Then there exist the generators $x_1, \cdots, x_{2n}, y_1, \cdots, y_m$ of G and a generator z of the derived subgroup of G, which satisfy*

$$\begin{array}{ll}
|x_i| = p^{t_i}, & i = 1, 2, \cdots, 2n, \\
|y_i| = p^{l_i}, & i = 1, 2, \cdots, m, \\
[x_{2i-1}, x_{2i}] = z, & i = 1, 2, \cdots, n, \\
[x_{2i-1}, x_j] = 1, & j \ne 2i, \\
[x_{2i}, x_j] = 1, & j \ne 2i-1, \\
[x_i^p, x_j] = 1, & i, j = 1, 2, \cdots, 2n, \\
[x_i, y_j] = 1, & i = 1, 2, \cdots, 2n, \ j = 1, 2, \cdots, m, \\
[y_i, y_j] = 1, & i, j = 1, 2, \cdots, m, \\
[x_i, z] = 1, & i = 1, 2, \cdots, 2n, \\
[y_i, z] = 1, & i = 1, 2, \cdots, m.
\end{array}$$

Theorem 1 *Let G be a finite p-group with derived subgroup of order p, and $|G/\zeta G| = p^{2n}$. Then*

(1) *if $p = 2$, then $nc(G) = 2n + 1$;*

(2) *if p is odd, then $np + 1 \le nc(G) \le (p(p-1)^n - 2)/(p-2)$.*

Proof. There exist the generators of G: $x_1, \cdots, x_{2n}, y_1, \cdots, y_m, z$, which satisfy the conditions in Lemma 1. Obviously, $\zeta G = \langle x_1^p, \cdots, x_{2n}^p, y_1, \cdots, y_m, z\rangle$.

Firstly, we assert

$$\begin{aligned}
&[(x_1^{r_1}x_2^{s_1})(x_3^{r_2}x_4^{s_2})\cdots(x_{2n-1}^{r_n}x_{2n}^{s_n})]\cdot[(x_1^{r_1'}x_2^{s_1'})(x_3^{r_2'}x_4^{s_2'})\cdots(x_{2n-1}^{r_n'}x_{2n}^{s_n'})] \\
={}&[(x_1^{r_1'}x_2^{s_1'})(x_3^{r_2'}x_4^{s_2'})\cdots(x_{2n-1}^{r_n'}x_{2n}^{s_n'})]\cdot[(x_1^{r_1}x_2^{s_1})(x_3^{r_2}x_4^{s_2})\cdots(x_{2n-1}^{r_n}x_{2n}^{s_n})]
\end{aligned}$$

if and only if

$$r_1s_1' + r_2s_2' + \cdots + r_ns_n' \equiv s_1r_1' + s_2r_2' + \cdots + s_nr_n' \pmod p,$$

where $0 \le r_i, r_i', s_i, s_i' < p$, and $i = 1, 2, \cdots, n$.

In fact, since

$$[x_{2i-1}, x_{2i}] = z, \qquad i = 1, 2, \cdots, n$$

and

$$x_{2i-1}^{r_i}x_{2i}^{s_i} = x_{2i}^{s_i}x_{2i-1}^{r_i}z^{r_is_i}, \qquad 0 \le r_i, s_i < p,$$

we have

$$\begin{aligned}
&[(x_1^{r_1}x_2^{s_1})(x_3^{r_2}x_4^{s_2})\cdots(x_{2n-1}^{r_n}x_{2n}^{s_n})]\cdot[(x_1^{r_1'}x_2^{s_1'})(x_3^{r_2'}x_4^{s_2'})\cdots(x_{2n-1}^{r_n'}x_{2n}^{s_n'})] \\
={}&[(x_1^{r_1'}x_2^{s_1'})(x_3^{r_2'}x_4^{s_2'})\cdots(x_{2n-1}^{r_n'}x_{2n}^{s_n'})]\cdot[(x_1^{r_1}x_2^{s_1})(x_3^{r_2}x_4^{s_2})\cdots(x_{2n-1}^{r_n}x_{2n}^{s_n})]
\end{aligned}$$