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Existence and Uniqueness of Positive Solutions for a System of Multi-order Fractional Differential Equations

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Abstract: In this paper, we prove the existence and uniqueness of positive solutions for a system of multi-order fractional differential equations. The system is used to represent constitutive relation for viscoelastic model of fractional differential equations. Our results are based on the fixed point theorems of increasing operator and the cone theory, some illustrative examples are also presented.

Key words: fractional differential equation, Caputo fractional derivative, fixed point theorem, positive solution

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1 Introduction

In recent years, fractional differential equations arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of complex medium, polymer rheology, economics, blood flow phenomena, etc., see [1]–[5].

Recently, the existence of solutions to fractional differential equations have been studied by many authors, see [6]–[14]. In this paper, we consider the following initial value problem for a system of fractional differential equations:

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$$\begin{cases} L_1(D)u(t) = f(t,\ D^\beta v(t)), & t \in [0,1], \\ L_2(D)v(t) = g(t,\ D^\alpha u(t)), & t \in [0,1], \\ u(0) = v(0) = 0, \end{cases} \tag{1.1}$$

where $L_1(D) = D^{S_n} - a_{n-1}D^{S_{n-1}} - \cdots - a_1 D^{S_1}$, $0 < \alpha < S_1 < \cdots < S_{n-1} < S_n \leq 1$, $a_i > 0$, $L_2(D) = D^{K_m} - b_{m-1}D^{K_{m-1}} - \cdots - b_1 D^{K_1}$, $0 < \beta < K_1 < \cdots < K_{m-1} < K_m \leq 1$, $b_j > 0$, $\alpha \leq K_m$, $\beta \leq S_n$, D^{S_i}, D^{K_j} $(i = 1, 2, \cdots, n-1;\ j = 1, 2, \cdots, m-1)$ are the standard Caputo fractional derivatives, and f, $g : [0,1] \times [0, +\infty) \to [0, +\infty)$ are continuous.

The system (1.1) is used to represent constitutive relation for viscoelastic model of fractional differential equations. Moreover, the system (1.1) describes processes of heat diffusion and combustion in two-component continua with nonlinear heat conduction and volumetric release. When we take Δu, Δv instead of u, v and let $S_1 = K_1 = n = m = 1$, it expresses as macroscopic models II for electrodiffusion of ions in nerve cells when molecular diffusion is anomalous subdiffusion due to binding, crowding or trapping (refer to [9]).

This paper is organized as follows. In Section 2, we present some basic materials needed to prove our main results. In Section 3, we prove the existence and uniqueness of positive solutions for (1.1) by applying the fixed point theorems of increasing operator and the cone theory.

2 Preliminaries

We recall several known definitions and properties from fractional calculus theory (refer to [1] and [15]).

Definition 2.1 *The Riemann-Liouville fractional integral of a function* $f : (0, 1] \to \mathbf{R}$ *of order* $\alpha > 0$ *is given by*

$$I_{0+}^\alpha f(t) := \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \mathrm{d}s,$$

provided that the right-hand side is pointwise defined on $(0, 1]$, *and* Γ *is the gamma function.*

Definition 2.2 *For at least* n*-times continuously differentiable function* $f : [0, +\infty) \to \mathbf{R}$, *the Caputo derivative of order* α *is defined as*

$$D_{0+}^\alpha f(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) \mathrm{d}s, \qquad n-1 < \alpha \leq n,\ t > 0,\ n \in \mathbf{N}.$$

Hereafter D^α denotes D_{0+}^α and I^α denotes I_{0+}^α.

Definition 2.3 *A Banach space* E *endowed with a closed cone* K *is an ordered Banach space* $(E,\ K)$ *with a partial order* $\leq$ *in* E *as follows*:

$$\boldsymbol{x} \leq \boldsymbol{y} \qquad \text{if } \boldsymbol{y} - \boldsymbol{x} \in K.$$

Definition 2.4 *For* $\boldsymbol{x}, \boldsymbol{y} \in E$, *the order interval* $\langle \boldsymbol{x},\ \boldsymbol{y} \rangle$ *is defined as*

$$\langle \boldsymbol{x},\ \boldsymbol{y} \rangle = \{\boldsymbol{z} \in E : \boldsymbol{x} \leq \boldsymbol{z} \leq \boldsymbol{y}\}.$$