Communications in Mathematical Research
31(1)(2015), 81–88

Constructing the Cocyclic Structures for Crossed Coproduct Coalgebras

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Communicated by Du Xian-kun

Abstract: In this paper, we construct a cocylindrical object associated to two coalgebras and a cotwisted map. It is shown that there exists an isomorphism between the cocyclic object of the crossed coproduct coalgebra induced from two coalgebras with a cotwisted map and the cocyclic object related to the diagonal of the cocylindrical object.

Key words: Hopf algebra, crossed coproduct coalgebra, cocyclic object
2010 MR subject classification: 16W30
Document code: A
Article ID: 1674-5647(2015)01-0081-08
DOI: 10.13447/j.1674-5647.2015.01.09

1 Introduction

Getzler and Jones[1] introduced a method to compute the cyclic homology of smash product algebras $A \rtimes G$, where G is a group that acts on an algebra A by automorphisms. This method is based on constructing a cylindrical object $A\sharp G$, and shows that $\varDelta(A\sharp G) \cong C_\bullet(A \rtimes G)$, where $\varDelta$ is the diagonal and $C_\bullet$ the cyclic object functor. Then by using the Eilenberg-Zilber theorem for cylindrical object, they obtained a quasi-isomorphism of mixed complexes $\varDelta(A\sharp G) \cong \mathrm{Tot}_\bullet(A\sharp G)$ and a spectral sequence converging to $HC_\bullet(A \rtimes G)$. This method has been used to computer the cyclic homology of some types of algebras in [2–4].

From the perspective of duality, it is necessary to consider the cocyclic structures of some coalgebras. It is the starting point of this paper to construct the cocyclic structures of crossed coproducts with invertible cotwisted maps, of which twisted smash coproducts in sense of Wang and Li[5] are special cases.

Received date: April 9, 2013.
Foundation item: The NSF (11261063, 11171183) of China, the Fund (2012ZDXK03) of the Key Disciplines in the General Colleges and Universities of Xinjiang Uygur Autonomous Region, the Foundation (2013721043) for Excellent Youth Science and Technology Innovation Talents of Xinjiang Uygur Autonomous Region and the NSF (ZR2011AM013) of Shandong Province.
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This paper is organized as follows: In Section 2, we recall the basic concepts of cocylindrical objects. Then crossed products with cotwisted maps are discussed in Section 3. The key content of this paper is to construct cocylindrical objects in Section 4. Finally, It is shown that there exists an isomorphism between the cocyclic object of the crossed coproduct coalgebra $A \times_T B$ and $\Delta(A\natural_T B)$ the cocyclic object related to the diagonal of $A\natural_T B$ In Section 5.

2 Cocylindrical Objects

Let us recall cocyclic objects. If $\mathcal{A}$ is any category, a paracocyclic object in $\mathcal{A}$ is a sequence of objects $\mathcal{A}_0, \mathcal{A}_1, \cdots$ together with coface operators $\partial^i : \mathcal{A}_n \to \mathcal{A}_{n+1}$ $(i = 0, 1, \cdots, n+1)$, codegeneracy operators, $\sigma^i : \mathcal{A}_n \to \mathcal{A}_{n-1}$ $(i = 0, 1, \cdots, n-1)$ and cyclic operators $\tau_n : \mathcal{A}_n \to \mathcal{A}_n$, where these operators satisfy the cosimplicial conditions and the following extra relations:

$$\tau_{n+1}\partial^i = \partial^{i-1}\tau_n, \quad 1 \le i \le n, \qquad \tau_{n+1}\partial^0 = \partial^n,$$
$$\tau_{n-1}\sigma^i = \sigma^{i-1}\tau_n, \quad 1 \le i \le n, \qquad \tau_{n-1}\sigma^0 = \sigma^n\tau_n^2.$$

If, in addition, $\tau_n^{n+1} = \mathrm{id}_n$, then we have a cocyclic object in the sense of Connes[6].

A bi-paracocyclic object in a category $\mathcal{A}$ is a double sequence $\mathcal{A}(p, q)$ of objects of $\mathcal{A}$ and operators $\partial_{p,q}$, $\sigma_{p,q}$, $\tau_{p,q}$ and $\bar{\partial}_{p,q}$, $\bar{\sigma}_{p,q}$, $\bar{\tau}_{p,q}$ such that for all $p \ge 0$, $q \ge 0$,

$$\mathcal{B}_p(q) = \{\mathcal{A}(p, q),\ \sigma^i_{p,q},\ \partial^i_{p,q},\ \tau_{p,q}\}, \qquad \mathcal{B}_q(p) = \{\mathcal{A}(p, q),\ \bar{\sigma}^i_{p,q},\ \bar{\partial}^i_{p,q},\ \bar{\tau}_{p,q}\}$$

are paracocyclic objects in $\mathcal{A}$ and every horizontal operator commutes with every vertical operator. We say that a bi-paracocyclic object is cocylindrical, if for all $p, q \ge 0$,

$$\bar{\tau}^{p+1}_{p,q}\tau^{q+1}_{p,q} = \mathrm{id}_{p,q}.$$

If A is a bi-paracocyclic object in a category $\mathcal{A}$, the paracocyclic object related to the diagonal of A is denoted by ΔA. So the paracocyclic operators on $\Delta A(n) = A(n, n)$ are $\bar{\partial}^i_{n,n+1}\partial^i_{n,n}$, $\bar{\sigma}^i_{n,n-1}\sigma^i_{n,n}$ and $\bar{\tau}^i_{n,n}\tau_{n,n}$, When A is a cocylindrical object, we conclude that $(\bar{\tau}_{n,n}\tau_{n,n})^{n+1} = \mathrm{id}_{n,n}$. So ΔA is a cocyclic object.

Throughout this paper, we works over a field k. All algebras and coalgebras are over k. The undecorated tensor product $\otimes$ means tensor product over k. Let C be a coalgebra. We use Sweedler's notation:

$\Delta(c) = c_{(1)} \otimes c_{(2)},\ \Delta^2(c) = (\Delta \otimes \mathrm{id})\Delta(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)},\ \cdots,\ \Delta^n c = (\mathrm{id} \otimes \Delta) \circ \Delta^{n-1}(c)$, where summation is omitted.

3 Crossed Coproducts with Cotwisted Maps

Let $(A,\ \Delta_A,\ \varepsilon_A)$ be a coalgebra and $(B,\ \Delta_B,\ \varepsilon_B)$ be a coalgebra. Give a linear map $T : A \otimes B \to B \otimes A$. Then $A \otimes B$ has the coproduct:

$$\Delta(a \otimes b) = a_{(1)} \otimes b_{(1)T} \otimes a_{(2)T} \otimes b_{(2)},$$

where $b_T \otimes a_T = b_U \otimes a_U = b_V \otimes a_V = b_X \otimes a_X = \cdots = T(a \otimes b)$ for all $b \in B,\ a \in A$. We say that $A \otimes B$ is a crossed coproduct which is denoted by $A \times_T B$, if $A \otimes B$ is a coalgebra with the counit $\varepsilon_A \otimes \varepsilon_B$. In this case, the map T is called a cotwisted map.