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On the Structure of the Units of Group Algebra of Dihedral Group

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Abstract: In this paper, we completely determine the structure of the unit group of the group algebra of some dihedral groups D_{2n} over the finite field F_{p^k}, where p is a prime.

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1 Introduction

Let F be a finite Galois field of prime characteristic p, G a finite group and FG the group algebra of the finite group G over the field F. Denote by $\mathcal{U}(FG)$ the unit group of FG. The main problems of the group algebra are decomposition of the group algebra. This suggests the related question: find all idempotent elements, nilpotent elements and invertible elements of FG. All these elements play an important role in the representation theory of the finite groups. Clearly, there exist techniques to find the decomposition of FG and the structure of FG when the characteristic of the field F does not divide the order of the group G. However, very little is known about $\mathcal{U}(FG)$ discussed in [1] when the characteristic of the field F is p and the order of the group is ap^m, where p is a prime, $(a, p) = 1$ and $a, m \in \mathbf{N}_0$. It is well known that if G is a finite p-group and F is a field of characteristic p, then $V(FG)$ is a finite p-group of order $|F|^{|G|-1}$, where $V(FG)$ is the subgroup of the unit group of FG consisting of the elements with augmentation 1. Let F_{p^k} be the Galois field of p^k elements. A basis for $V(F_pG)$ is determined, where F_p is the Galois field of p-elements and G is an Abelian p-group (see [2]). A basis for $V_*(FG)$ is established when F is any field of characteristic p and G is an Abelian p-group (see [3]), where $V_*(FG)$ are the unitary units of $V(FG)$. Also in [3],

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there are conditions provided when $V_*(FG)$ is normal in $V(FG)$. Let D_{2n} be the dihedral group of order $2n$. The order of $\mathcal{U}(F_{p^k}D_{2p^m})$ is determined and the structure of $\mathcal{U}(F_{3^k}D_6)$, $\mathcal{U}(F_{5^k}D_{10})$ and $\mathcal{U}(F_{5^k}D_{20})$ have been established in terms of split extensions of elementary Abelian groups. But the relationship among $\mathcal{U}(F_{p^k}C_n)$, $\mathcal{U}(F_{p^k}C_2)$ and $\mathcal{U}(F_{p^k}D_{2n})$ is unclear. In this paper, we give some messages on the relationship among $\mathcal{U}(F_{p^k}C_p)$, $\mathcal{U}(F_{p^k}C_2)$ and $\mathcal{U}(F_{p^k}D_{2p})$ and the relationship among $\mathcal{U}(F_{2^k}C_{2^n})$, $\mathcal{U}(F_{2^k}C_2)$ and $\mathcal{U}(F_{2^k}D_{2^{n+1}})$. Also, the structure of $\mathcal{U}(F_{2^k}D_{2^{n+1}})$ and $\mathcal{U}(F_{p^k}D_{2p})$ are determined.

2 Preliminaries

In this section, we mainly introduce some elementary knowledge and notations used in this paper.

Definition 2.1[4] *A circulant matrix over a field F is a square $n\times n$ matrix which has the form*

$$\operatorname{circ}(a_1, a_2, \cdots, a_n) = \begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_n & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_n & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & & \vdots \\ a_2 & a_3 & a_4 & \cdots & a_1 \end{pmatrix}, \qquad a_i \in F.$$

It is easy to see that all the $n \times n$ circulant matrices over a field F form a commutative ring and the inverses of the circulant matrices are also circulant.

If $G = \{g_1, g_2, \cdots, g_n\}$ is a fixed listing of the elements of a group G, then the matrix

$$\begin{pmatrix} g_1^{-1}g_1 & g_1^{-1}g_2 & g_1^{-1}g_3 & \cdots & g_1^{-1}g_n \\ g_2^{-1}g_1 & g_2^{-1}g_2 & g_2^{-1}g_3 & \cdots & g_2^{-1}g_n \\ g_3^{-1}g_1 & g_3^{-1}g_2 & g_3^{-1}g_3 & \cdots & g_3^{-1}g_n \\ \vdots & \vdots & \vdots & & \vdots \\ g_n^{-1}g_1 & g_n^{-1}g_2 & g_n^{-1}g_3 & \cdots & g_n^{-1}g_n \end{pmatrix}$$

is referred to as the matrix of G (relative to this listing) and denoted by $M(G)$. Let

$$\omega = \sum_{i=1}^{n} \alpha_{g_i} g_i \in FG,$$

where F is a field and $\alpha_{g_i} \in F$. Then the matrix

$$\begin{pmatrix} \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \alpha_{g_1^{-1}g_3} & \cdots & \alpha_{g_1^{-1}g_n} \\ \alpha_{g_2^{-1}g_1} & \alpha_{g_2^{-1}g_2} & \alpha_{g_2^{-1}g_3} & \cdots & \alpha_{g_2^{-1}g_n} \\ \alpha_{g_3^{-1}g_1} & \alpha_{g_3^{-1}g_2} & \alpha_{g_3^{-1}g_3} & \cdots & \alpha_{g_3^{-1}g_n} \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{g_n^{-1}g_1} & \alpha_{g_n^{-1}g_2} & \alpha_{g_n^{-1}g_3} & \cdots & \alpha_{g_n^{-1}g_n} \end{pmatrix}$$