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Interpolation by Bivariate Polynomials Based on Multivariate F-truncated Powers

Yuan Xue-mei

(*General Courses Department, Academy of Military Transportation, Tianjin, 300160*)

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Abstract: The solvability of the interpolation by bivariate polynomials based on multivariate F-truncated powers is considered in this short note. It unifies the point-wise Lagrange interpolation by bivariate polynomials and the interpolation by bivariate polynomials based on linear integrals over segments in some sense.

Key words: multivariate F-truncated power, point-wise Lagrange interpolation, solvability of an interpolation problem

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1 Introduction

Suppose that $\boldsymbol{M}$ is an $s\times n$ real matrix with $\text{rank}(\boldsymbol{M}) = s$ and $f(x_1, \cdots, x_n)$ is an n-variables real function defined on

$$\mathbf{R}^n_+ := \{\boldsymbol{x} = (x_1, \cdots, x_n) \in \mathbf{R}^n \mid x_i \geq 0,\ i = 1, \cdots, n\}.$$

The multivariate F-truncated power $T_f(\,\cdot\,|\boldsymbol{M})$ associated with $\boldsymbol{M}$ and f is defined as in [1]:

$$\int_{\mathbf{R}^s} T_f(\boldsymbol{x}|\boldsymbol{M})\phi(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = \int_{\mathbf{R}^n_+} f(\boldsymbol{u})\phi(\boldsymbol{M}\boldsymbol{u})\mathrm{d}\boldsymbol{u}, \qquad \phi \in \mathscr{D}(\mathbf{R}^s), \tag{1.1}$$

where $\mathscr{D}(\mathbf{R}^s)$ is the space of test functions on $\mathbf{R}^s$, i.e., the space of all compactly supported and infinitely differentiable functions on $\mathbf{R}^s$.

Based on (1.1), one can conclude that (see [1])

$$T_f(\boldsymbol{x}|\boldsymbol{M}) = \frac{1}{\sqrt{\det(\boldsymbol{M}\boldsymbol{M}^{\mathrm{T}})}} \int_{\boldsymbol{M}\boldsymbol{u}=\boldsymbol{x},\, \boldsymbol{u}\in\mathbf{R}^n_+} f(\boldsymbol{u})\mathrm{d}\mu, \tag{1.2}$$

where μ is the Lebesgue measure on the $(n-s)$-dimensional affine variety $\mathcal{H}$ that contains

$$\{\boldsymbol{u} \mid \boldsymbol{M}\boldsymbol{u} = \boldsymbol{x},\ \boldsymbol{u} \in \mathbf{R}^n_+\}.$$

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E-mail address: yuanxuemei_8013@163.com (Yuan X M).

Based on (1.2), one can see that $T_f(\boldsymbol{x}|\boldsymbol{M})$ is linear with respect to f, that is,

$$T_{f_1+f_2}(\boldsymbol{x}|\boldsymbol{M}) = T_{f_1}(\boldsymbol{x}|\boldsymbol{M}) + T_{f_2}(\boldsymbol{x}|\boldsymbol{M})$$

and

$$T_{\lambda f}(\boldsymbol{x}|\boldsymbol{M}) = \lambda T_f(\boldsymbol{x}|\boldsymbol{M}), \qquad \lambda \in \mathbf{R}.$$

Moreover, if $f \equiv 1$, then

$$T_f(\boldsymbol{x}|\boldsymbol{M}) = T(\boldsymbol{x}|\boldsymbol{M})$$

is reduced to the classical multivariate truncated power.

We next turn to an interpolation problem. We use Π_l^n to denote the n-variables polynomial space of total degrees no larger than l. Given a set of pairs $\{(\boldsymbol{x}^{(i)},\ \boldsymbol{M}^{(i)})\}_{i=1}^N$, where $\boldsymbol{M}^{(i)}$ are $s \times n$ real matrices, $\boldsymbol{x}^{(i)} \in \mathbf{R}^s$, and $N = \begin{pmatrix} n+l \\ n \end{pmatrix}$. We say that it is poised in Π_l^n, if for all $\{\gamma_i\}_{i=1}^N \subset \mathbf{R}$ there exists a unique $P \in \Pi_l^n$ such that

$$T_P(\boldsymbol{x}^{(i)}|\boldsymbol{M}^{(i)}) = \gamma_i, \qquad i = 1, 2, \cdots, N. \tag{1.3}$$

The interpolation problem (1.3) is called the interpolation by multivariate polynomials based on multivariate F-truncated powers. If $\{(\boldsymbol{x}^{(i)},\ \boldsymbol{M}^{(i)})\}_{i=1}^N$ is poised in Π_l^n, then (1.3) is called to be solvable. It is easy to see that $\{(\boldsymbol{x}^{(i)},\ \boldsymbol{M}^{(i)})\}_{i=1}^N$ is poised in Π_l^n if and only if

$$T_P(\boldsymbol{x}^{(i)}|\boldsymbol{M}^{(i)}) = 0, \qquad i = 1, 2, \cdots, N,$$

which implies $P \equiv 0$.

In this short note, we only consider the solvability of (1.3) for the cases $n = 2$ and $s = 1, 2$. Sufficient and necessary conditions are obtained to guarantee the set of pairs $\{(\boldsymbol{x}^{(i)},\ \boldsymbol{M}^{(i)})\}_{i=1}^N$ to be poised. Referring to the point-wise Lagrange interpolation by bivariate polynomials (see [2–3]) and the interpolation by bivariate polynomials based on linear integrals over segments (see [4–6]), for the case that $n = 2$ and $s = 2$, (1.3) is a point-wise Lagrange interpolation by bivariate polynomials and for the case that $n = 2$ and $s = 1$ is an interpolation by bivariate polynomials based on linear integrals. Therefore we can say that we unify the point-wise Lagrange interpolation and the interpolation based on linear integrals to the interpolation based on multivariate F-truncated powers. Our main results are stated as follows.

Theorem 1.1 *Suppose that $n = 2$ and $s = 2$. The set of pairs $\{(\boldsymbol{x}^{(i)},\ \boldsymbol{M}^{(i)})\}_{i=1}^N$ is poised in Π_l^2 if and only if*

$$\boldsymbol{x}^{(i)} \in \operatorname{cone}(\boldsymbol{M}^{(i)}), \qquad i = 1, 2, \cdots, N,$$

and $\{(\boldsymbol{M}^{(i)})^{-1}(\boldsymbol{x}^{(i)})\}_{i=1}^N$ is poised for the point-wise Lagrange interpolation in Π_l^2, where

$$\operatorname{cone}(\boldsymbol{M}^{(i)}) = \left\{ \sum_{j=1}^{2} u_j \boldsymbol{m}_{ij} \mid (u_1, u_2) \in \mathbf{R}_+^2,\ \boldsymbol{M}^{(i)} = (\boldsymbol{m}_{i1},\ \boldsymbol{m}_{i2}) \right\},$$

and $\boldsymbol{m}_{ij}$ $(j = 1, 2)$ are the column vectors of $\boldsymbol{M}^{(i)}$.

Theorem 1.2 *Suppose that $n = 2$ and $s = 1$. The set of pairs $\{(\boldsymbol{x}^{(i)},\ \boldsymbol{M}^{(i)})\}_{i=1}^N$ is poised in Π_l^2 if and only if*

$$\boldsymbol{x}^{(i)} \in \operatorname{cone}(\boldsymbol{M}^{(i)}), \quad \boldsymbol{m}_{i1} \cdot \boldsymbol{m}_{i2} > 0, \qquad \boldsymbol{x}^{(i)} \neq 0,\ i = 1, 2, \cdots, N,$$

and $\left\{ \left(\dfrac{\boldsymbol{x}^{(i)}}{\boldsymbol{m}_{i1}},\ \dfrac{\boldsymbol{x}^{(i)}}{\boldsymbol{m}_{i2}} \right) \right\}_{i=1}^N$ is poised for the point-wise Lagrange interpolation in Π_l^2.