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A Class of Metric Spaces Which Do Not Coarsely Contain Expanders

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Abstract: In this paper, a class of metric spaces which include Hilbert spaces and Hadamard manifolds are defined. And the expanders cannot be coarsely embedded into this class of metric spaces are proved.

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Let X, Y be two metric spaces. A map $f : X \to Y$ is called a coarse embedding if there exist two non-decreasing functions ρ_1, $\rho_2 : [0, \infty) \to [0, \infty)$ such that

$$\lim_{t\to\infty} \rho_1(t) = \infty$$

and

$$\rho_1(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_2(d_X(x, y)), \qquad x, y \in X.$$

If X admits a coarse embedding into Y, then X is called coarsely embeddable into Y or Y coarsely contains X.

In [1], it was proven that the coarse Baum-Connes conjecture holds for spaces which admit a coarse embedding into Hilbert spaces. Similarly in [2–3], it was proven that the coarse geometric Novikov conjecture holds for spaces which admit a coarse embedding into Hadamard manifolds. On the other hand, it is also known that there exist spaces which cannot be coarsely embedded into either Hilbert spaces or Hadamard manifolds. The known construction of such coarsely non-embeddable spaces depends on expanders.

Let (V, E) be a finite graph with the vertex set V and the edge set E. We denote the cardinality of V and E by $|V|$ and $|E|$, respectively. We also define an orientation on E.

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The differential $d : \ell_2(V) \to \ell_2(E)$ is defined by

$$d(f)(e) = f(e^+) - f(e^-), \qquad f \in \ell_2(V),\ e \in E$$

with the starting vertex e^+ and the ending vertex e^-.

The Laplace operator $\Delta = d \star d$, where $d\star$ is the adjoint operator of d. This definition does not depend on the choice of the orientation of E. Apparently, Δ is self-adjoint. Also it is positive since $\langle \Delta f, f\rangle = \langle df, df\rangle \geq 0$. Hence Δ has real nonnegative eigenvalues. We denote $\lambda_1(V)$ the minimal positive eigenvalue of the Laplace operator Δ on the graph (V, E).

Definition 1 *A sequence of graphs $\{(V_n, E_n)\}_{n=1}^{\infty}$ of the fixed degree l and with $|V_n|$ approaching to ∞ is called an expander if there is a positive constant c such that $\lambda_1(V_n) \geq c$ for all $n \in \mathbf{N}^+$. The largest possible c is called the Laplace constant of $\{(V_n, E_n)\}_{n=1}^{\infty}$.*

To say that an expander $\{(V_n, E_n)\}_{n=1}^{\infty}$ is coarsely embeddable into a metric space X, we mean that there exist two non-decreasing functions $\rho_1, \rho_2 : [0, \infty) \to [0, \infty)$ such that for all V_n,

$$\rho_1(d_{V_n}(x, y)) \leq d_X(f(x), f(y)) \leq \rho_2(d_{V_n}(x, y)), \qquad x, y \in V_n$$

and

$$\lim_{t\to\infty} \rho_1(t) = \infty.$$

As we mentioned at the beginning,

Theorem 1 *An expander is not coarsely embeddable into Hilbert spaces and Hadamard manifolds.*

The proofs of the coarse non-embeddability of expanders into Hilbert spaces and Hadamard manifolds are different from literature (cf. [4–6]). Here we define a property and unify the proof for both cases. Let $B_R^X(x) = \{y \in X \mid d(x, y) < R\}$.

Definition 2 *A metric space X is called special if there exists a family of s-Lipschitz maps $\{f_x : X \to \mathcal{H} \mid x \in X\}$, where $\mathcal{H}$ is a Hilbert space and $R, l > 0$ such that*

(1) *for any finite subset $A \subset X$, there exists an $x_A \in X$ such that $\sum\limits_{x\in A} f_{x_A}(x) = 0$;*

(2) *for any $K > 0$, there exists an $L > 0$ such that for any $x \in X$, there exists an $x_0 \in X$ and $f_x^{-1}(B_K^{\mathcal{H}}(0)) \subset B_L^X(x_0)$.*

Example 1 Hilbert spaces and Hadamard manifolds are special.

(1) Let $\mathcal{H}$ be a Hilbert space and $x \in \mathcal{H}$, we define $f_x : \mathcal{H} \to \mathcal{H}$ as $f_x(y) = y - x$ for all $y \in \mathcal{H}$. Then f_x is 1-Lipschitz. Given a finite subset $A \subset \mathcal{H}$, let $x_0 = \dfrac{1}{|A|}\sum\limits_{x\in A} x$. Then

$$\sum_{x\in A} f_{x_0}(x) = \sum_{x\in A}(x - x_0) = 0.$$

Clearly, it satisfies all conditions with a choice $L = K$.

(2) For a Hadamard manifold $\mathcal{M}$, we present Higson's argument. This is the origin of the definition of the special property. Assume that $\dim \mathcal{M} = m$. Let A be a finite subset of