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The Closed Subsemigroups of a Clifford Semigroup

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Abstract: In this paper we study the closed subsemigroups of a Clifford semigroup. It is shown that $\left\{ \bigcup\limits_{\alpha \in \overline{Y'}} G_\alpha \mid Y' \in P(Y) \right\}$ is the set of all closed subsemigroups of a Clifford semigroup $S = [Y; G_\alpha; \phi_{\alpha,\beta}]$, where $\overline{Y'}$ denotes the subsemilattice of Y generated by Y'. In particular, G is the only closed subsemigroup of itself for a group G and each one of subsemilattices of a semilattice is closed. Also, it is shown that the semiring $\overline{P}(S)$ is isomorphic to the semiring $\overline{P}(Y)$ for a Clifford semigroup $S = [Y; G_\alpha; \phi_{\alpha,\beta}]$.

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1 Introduction

By a semiring we mean a type $\langle 2, 2\rangle$ algebra $(S, +, \cdot)$ satisfying the following identities:

(SR1) $x + (y + z) \approx (x + y) + z$;

(SR2) $x(yz) \approx (xy)z$;

(SR3) $x(y + z) \approx xy + xz$, $(x + y)z = xz + yz$.

The power semiring of a semigroup S and the closed subsemigroups of S are introduced and studied by Zhao[1]. By studying of the power semiring of an idempotent semigroup S and the closed subsemigroups of an idempotent semigroup S, in [2–3], Pastijn *et al.* obtained the lattice of all subvarieties of the variety consisting of the semirings S for which $(S, +)$ is a semilattice and $(S, \cdot)$ is an idempotent semigroup (the concepts of lattices and varieties

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are introduced in [4]). This lattice is distributive and contains 78 varieties precisely. Each of those is finitely based and generated by a finite number of finite ordered bands.

Let S be a semigroup and $P(S)$ the set of all nonempty subsets of S. For $A, B \in P(S)$, we define

$$A \circ B = \{ab \mid a \in A,\ b \in B\}.$$

Then $(P(S), \cup, \circ)$ becomes a semiring, which is called the power semiring of S. A subsemigroup C of a semigroup S is said to be closed (see [1]) if

$$sat, sbt \in C \Rightarrow sabt \in C$$

holds for all $a, b \in S,\ s, t \in S^1$. The set of all closed subsemigroups of S is denoted by $\overline{P}(S)$.

Let S be a semigroup and A a nonempty subset of S. $\overline{A}$ denotes the closed subsemigroup of S generated by A, i.e., the smallest closed subsemigroup of S containing A. Define inductively (see [1]) sets $A^{(i)}$ $(i \geq 1)$ as follows: $A^{(1)}$ is the subsemigroup of S generated by A; for any $i \geq 1$, $A^{(i+1)}$ is the subsemigroup of S generated by the set

$$A^{(i)} \bigcup \{scdt \mid sct, sdt \in A^{(i)},\ c, d \in S,\ s, t \in S^1\}.$$

Zhao[1] proved that $\overline{A} = \bigcup\limits_{i \geq 1} A^{(i)}$.

Let S be a semigroup and $\overline{P}(S)$ the set of all closed subsemigroups of S. Then $\overline{P}(S)$ becomes a semiring equipped with the addition and the multiplication as follows:

$$A + B = \overline{A \cup B}, \qquad AB = \overline{A \circ B}.$$

Also, it is easy to see that the mapping

$$\tau : P(S) \longrightarrow \overline{P}(S), \qquad A \longmapsto \overline{A}$$

is a semiring homomorphism. The kernel of τ was written as ρ in [1]. That is to say,

$$A \rho B \Leftrightarrow \overline{A} = \overline{B}, \qquad A, B \in \overline{P}(S).$$

Clifford[5] introduced Clifford semigroups which play an important role in the theory of semigroups. In this paper we study the closed subsemigroups of a Clifford semigroup. It is shown that $\left\{ \bigcup\limits_{\alpha \in \overline{Y'}} G_\alpha \mid Y' \in P(Y) \right\}$ is the set of all closed subsemigroups of a Clifford semigroup $S = [Y; G_\alpha; \phi_{\alpha,\beta}]$, where $\overline{Y'}$ denotes the subsemilattice of Y generated by Y'. In particular, G is the only closed subsemigroup of itself for a group G and each one of subsemilattices of a semilattice is closed. Also, it is shown that the semiring $\overline{P}(S)$ is isomorphic to the semiring $\overline{P}(Y)$ for a Clifford semigroup $S = [Y; G_\alpha; \phi_{\alpha,\beta}]$.

2 Closed Subsemigroup

Theorem 2.1 *Let G be a group and* 1 *the identity element of G. Then*

$$\overline{A} = G, \qquad A \in P(G).$$

Proof. We first prove that $\overline{\{1\}} = G$. It is obvious that $\overline{\{1\}} \subseteq G$. Also, since $1 = 1aa^{-1} \in \overline{\{1\}}$ for any $a \in G$, we have $a = 1aaa^{-1} \in \overline{\{1\}}$ by the definition of closed subsemigroups. That is to say, $G \subseteq \overline{\{1\}}$. So it follows that $\overline{\{1\}} = G$.