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Blow-up Sets to a Coupled Heat System

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Abstract: This paper deals with a heat system coupled via local and localized sources subject to null Dirichlet boundary conditions. In a previous paper of the authors, a complete result on the multiple blow-up rates was obtained. In the present paper, we continue to consider the blow-up sets to the system via a complete classification for the nonlinear parameters. That is the discussion on single point versus total blow-up of the solutions. It is mentioned that due to the influence of the localized sources, there is some substantial difficulty to be overcomed there to deal with the single point blow-up of the solutions.

Key words: coupled localized source, coupled local source, total blow-up, single point blow-up, blow-up set

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1 Introduction

This paper considers the following heat system coupled via local and localized sources:

$$\begin{cases} u_t = \Delta u + v^{p_1} + v^{q_1}(0,t), & (x,t) \in B \times (0,T), \\ v_t = \Delta v + u^{p_2} + u^{q_2}(0,t), & (x,t) \in B \times (0,T), \\ u = v = 0, & (x,t) \in \partial B \times (0,T), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \bar{B}, \end{cases} \tag{1.1}$$

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where $B = \{x \in \mathbf{R}^N : |x| < 1\}$, $p_1, p_2 > 1$, $q_1, q_2 > 0$, the radial initial data $u_0, v_0 \in C^2(B) \cap C(\bar{B})$ satisfy

$$\text{(A)} \qquad \begin{cases} u_0 = u_0(r), \quad v_0 = v_0(r), & u_0, v_0 \geq 0, \ u_0(0), \ v_0(0) > 1, \\ u_0(1) = v_0(1) = 0, & u_{0r}, v_{0r} < 0 \text{ for } r \in (0, 1], \end{cases}$$

and

$$\text{(B)} \qquad \begin{cases} \Delta u_0 + v_0^{p_1} + v_0^{q_1}(0) \geq \eta\varphi_0(v_0^{p_1} + v_0^{q_1}(0), & x \in \bar{B}, \\ \Delta v_0 + u_0^{p_2} + u_0^{q_2}(0) \geq \eta\varphi_0(u_0^{p_2} + u_0^{q_2}(0)), & x \in \bar{B}, \end{cases}$$

where $\eta \in \Big(0, \dfrac{1}{2}\Big]$, λ_0 and $\varphi_0 \in C^2(B) \cap C(\bar{B})$ are the first eigenvalue and eigenfunction, respectively, of

$$\begin{cases} \Delta\varphi + \lambda\varphi = 0 & \text{in } B, \\ \varphi = 0 & \text{on } \partial B \end{cases} \tag{1.2}$$

normalized by $\varphi_0 > 0$ in B and $\|\varphi_0\|_\infty = 1$. Obviously, φ_0 is a radially symmetric function with $\varphi_0' < 0$ for $r \in (0, 1]$. Such u_0 and v_0 do exist indeed (see [1–2]).

The theory of parabolic equations insures that there exists a unique local solution to (1.1), which blows up in finite time for large initial data, (see, e.g., [3–5]). Let T be the maximum existence time of the solution.

The system (1.1) is a combination of the following two coupled problems: with local coupling

$$u_t = \Delta u + v^{p_1}, \quad v_t = \Delta v + u^{p_2}, \qquad (x, t) \in \Omega \times (0, T), \tag{1.3}$$

and with localized coupling

$$u_t = \Delta u + v^{q_1}(0, t), \quad v_t = \Delta v + u^{q_2}(0, t), \qquad (x, t) \in \Omega \times (0, T), \tag{1.4}$$

subject to null Dirichlet boundary conditions, where Ω is a bounded domain.

It is well known that the blow-up solutions of (1.3) with $p_1p_2 > 1$ must be single point blow-up (see [5–7]). In [5] the single point blow-up result was proved for $n = 1$ with a very restrictive condition of $p_1 = p_2$. This restriction of $p_1 = p_2$ was removed by Souplet[7], which is a substantially improvement for the single blow-up discussion. On the other hand, we know that the blow-up occurs everywhere in $\Omega = B$ for (1.4) with $q_1q_2 > 1$ (see [8]). Naturally, both total and single point blow-up may be possible for (1.1).

The total versus single point blow-up for the scalar equation with both local and localized sources

$$u_t = \Delta u + u^p(x, t) + u^q(x^*, t), \qquad (x, t) \in B \times (0, T)$$

was well studied by Okada *et al.*[9–10] As for system, little is known concerning the total and single point blow-up (see [1, 11]).

This paper is arranged as follows: The next section gives the multiple blow-up rate results obtained in [12] as the preliminaries of the paper. Sections 3 and 4 are devoted to the discussion on total and single point blow-up, respectively.