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Cocycle Perturbation on Banach Algebras

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Abstract: Let α be a flow on a Banach algebra $\mathfrak{B}$, and $t \mapsto u_t$ a continuous function from $\mathbf{R}$ into the group of invertible elements of $\mathfrak{B}$ such that $u_s\alpha_s(u_t) = u_{s+t}$, $s, t \in \mathbf{R}$. Then $\beta_t = \mathrm{Ad}u_t \circ \alpha_t$, $t \in \mathbf{R}$ is also a flow on $\mathfrak{B}$, where $\mathrm{Ad}u_t(B) \triangleq u_t B u_t^{-1}$ for any $B \in \mathfrak{B}$. β is said to be a cocycle perturbation of α. We show that if α, β are two flows on a nest algebra (or quasi-triangular algebra), then β is a cocycle perturbation of α. And the flows on a nest algebra (or quasi-triangular algebra) are all uniformly continuous.

Key words: cocycle perturbation, inner perturbation, nest algebra, quasi-triangular algebra

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1 Introduction

In the quantum mechanics of particle systems with an infinite number of degrees of freedom, an important problem is to study the differential equation

$$\frac{\mathrm{d}\alpha_t(A)}{\mathrm{d}t} = S\alpha_t(A)$$

under variety of circumstances and assumptions. In each instance the A corresponds to an observable, or state, of the system and is represented by an element of some suitable Banach algebra $\mathfrak{B}$. S is an operator on $\mathfrak{B}$, and $\{\alpha_t\}_{t\in\mathbf{R}}$ is a group of bounded automorphisms on $\mathfrak{B}$. The function

$$t \in \mathbf{R} \mapsto \alpha_t(A) \in \mathfrak{B}$$

describes the motion of A. The dynamics are given by solutions of the differential equation subject to certain supplementary conditions of continuity. Thus it is worth to study the group of bounded automorphisms on $\mathfrak{B}$. For more details see [1].

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A flow α on $\mathfrak{B}$ is a group homomorphism of the real line $\mathbf{R}$ into the group of bounded automorphisms on $\mathfrak{B}$ (i.e., $t \mapsto \alpha_t$) such that

$$\lim_{t \to t_0} \|\alpha_t(B) - \alpha_{t_0}(B)\| = 0, \qquad t_0 \in \mathbf{R},\ B \in \mathfrak{B}.$$

If there exists an $h \in \mathfrak{B}$ such that

$$\alpha_t(B) = \mathrm{e}^{th} B \mathrm{e}^{-th}, \qquad B \in \mathfrak{B},\ t \in \mathbf{R},$$

then we call α an inner flow. We say that a flow α is uniformly continuous if

$$\lim_{t \to t_0} \|\alpha_t - \alpha_{t_0}\| = 0, \qquad t_0 \in \mathbf{R}.$$

If α is a flow on $\mathfrak{B}$ and if u is a continuous map of $\mathbf{R}$ into the group of invertible elements $G(\mathfrak{B})$ of $\mathfrak{B}$ such that

$$u_s \alpha_s(u_t) = u_{s+t}, \qquad s, t \in \mathbf{R},$$

then we call $u = (u_t)_{t \in \mathbf{R}}$ an α-cocycle for $(\mathfrak{B},\ \mathbf{R},\ \alpha)$. Let

$$\beta_t = \mathrm{Ad}u_t \circ \alpha_t, \qquad t \in \mathbf{R},$$

where

$$\mathrm{Ad}u_t(B) \triangleq u_t B u_t^{-1},$$

i.e.,

$$\beta_t(B) = u_t \alpha_t(B) u_t^{-1}, \qquad B \in \mathfrak{B}.$$

Then β is also a flow on $\mathfrak{B}$, and is said to be a cocycle perturbation of α.

If α is a flow on $\mathfrak{B}$, let $D(\delta_\alpha)$ be composed of those $B \in \mathfrak{B}$ for which there exists an $A \in \mathfrak{B}$ with the property that

$$A = \lim_{t \to 0} \frac{\alpha_t(B) - B}{t}.$$

Then δ_α is a linear operator on $D(\delta_\alpha)$ defined by

$$\delta_\alpha(B) = A.$$

We call δ_α the infinitesimal generator of α. By Proposition 3.1.6 of [1], δ_α is a closed derivation, i.e., the domain $D(\delta_\alpha)$ is a dense subalgebra of $\mathfrak{B}$ and δ_α is closed as a linear operator on $D(\delta_\alpha)$ and satisfies

$$\delta_\alpha(AB) = \delta_\alpha(A)B + A\delta_\alpha(B), \qquad A, B \in D(\delta_\alpha).$$

We call β an inner perturbation of α if α, β are two flows on $\mathfrak{B}$,

$$D(\delta_\alpha) = D(\delta_\beta),$$

and there exists an $h \in \mathfrak{B}$ such that

$$\delta_\beta = \delta_\alpha + \mathrm{adi}h,$$

where i is the imaginary unit, and

$$\mathrm{adi}h(B) \triangleq \mathrm{i}(hB - Bh), \qquad B \in \mathfrak{B}.$$

Moreover,

$$D(\delta_\alpha) = \mathfrak{B}$$

if and only if α is uniformly continuous. For more details see [1–2].

The problem we consider here is classifying cocycle of flows on Banach algebras. Such a problem has been considered in the C^*-algebra cases, notably by Kishimoto[3–10]. We refer the reader to [3] for a detailed study of the general results concerning cocycles and invariants