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Cofiniteness of Local Cohomology Modules with Respect to a Pair of Ideals

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Abstract: Let R be a commutative Noetherian ring, I and J be two ideals of R, and M be an R-module. We study the cofiniteness and finiteness of the local cohomology module $H^i_{I,J}(M)$ and give some conditions for the finiteness of $\mathrm{Hom}_R(R/I,\ H^s_{I,J}(M))$ and $\mathrm{Ext}^1_R(R/I,\ H^s_{I,J}(M))$. Also, we get some results on the attached primes of $H^{\dim M}_{I,J}(M)$.

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1 Introduction

Throughout this paper, we always assume that R is a commutative Noetherian ring, I and J are two ideals of R, and M is an R-module. Takahashi *et al.*[1] introduced the concept of local cohomology module $H^i_{I,J}(M)$ with respect to a pair of ideals (I, J). The set of elements x of M such that $I^n \subseteq \mathrm{Ann}(x) + J$ for some integer $n \gg 1$ is said to be (I, J)-torsion submodule of M and is denoted by $\Gamma_{I,J}(M)$. For an integer $i \geq 0$, the local cohomology functor $H^i_{I,J}$ with respect to (I, J) is defined to be the i-th right derived functor of $\Gamma_{I,J}$. Note that, if $J = 0$, then $H^i_{I,J}(\cdot)$ coincides with $H^i_I(\cdot)$. When M is finitely generated, we know that $H^i_{I,J}(M) = 0$ for $i > \dim M$ from Theorem 4.7 in [1].

Hartshorne[2] defined an R-module M to be I-cofinite if

$$\mathrm{Supp} M \subseteq V(I) \qquad \text{and} \qquad \mathrm{Ext}^i_R(R/I,\ M)$$

is finitely generated for all $i \geq 0$. Also, he asked the following question:

Question If M is finitely generated, when is $\mathrm{Ext}^j_R(R/I,\ H^i_I(M))$ finitely generated for

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all $i \geq 0$ and $j \geq 0$ (considering $\mathrm{Supp}(H_I^i(M)) \subseteq V(I)$, so $\mathrm{Ext}_R^j(R/I,\ H_I^i(M))$ is finitely generated if and only if $H_I^i(M)$ is I-cofinite).

Hartshorne[2] showed that if $(R, \mathfrak{m})$ is a complete regular local ring and M is finitely generated, then $H_I^i(M)$ is I-cofinite in two cases:

(a) I is a non-zero principal ideal;

(b) I is a prime ideal with $\dim R/I = 1$.

Yoshida[3], Delfino and Marley[4] extended (b) to all dimension one ideals I of any local ring R, and Kawasaki[5] proved (a) for any ring R.

Let

$$W(I, J) = \{p \in \mathrm{Spec}(R) \mid I^n \subseteq J + p \text{ for an integer } n \gg 1\}.$$

As a generalization of I-cofinite module, we give the following definition:

Definition 1.1 *An R-module M is said to be (I, J)-cofinite if* $\mathrm{Supp}M \subseteq W(I, J)$ *and* $\mathrm{Ext}_R^i(R/I,\ M)$ *is finitely generated for all $i \geq 0$.*

For an R-module M, the cohomological dimension of M with respect to I and J is defined as

$$\mathrm{cd}(I, J, M) = \sup\{i \in \mathbf{Z} \mid H_{I,J}^i(M) \neq 0\}.$$

When $J = 0$, then $\mathrm{cd}(I, J, M)$ coincides with $\mathrm{cd}(I, M)$.

In this paper, we mainly consider the (I, J)-cofiniteness of $H_{I,J}^i(M)$. Since

$$\mathrm{Supp}(H_{I,J}^i(M)) \subseteq W(I, J),$$

we focus on the finiteness of $\mathrm{Ext}_R^j(R/I,\ H_{I,J}^i(M))$.

In Section 2, we discuss the finiteness of $\mathrm{Hom}_R(R/I,\ H_{I,J}^s(M))$ (see Theorem 2.1), which generalizes Theorem 2.1 in [6] and Theorem B(β) in [7]. In addition, when M is finitely generated and I is a principal ideal or $\mathrm{cd}(I, J, M) = 1$, we get the (I, J)-cofiniteness of $H_{I,J}^i(M)$ for all $i \geq 0$, which generalizes the corresponding results in [5] and [9], respectively. In Proposition 2.3(iii) of [10], it is proved that if

$$H_I^i(M) = 0, \qquad 0 \leq i < s,$$

then

$$\mathrm{Hom}(R/I,\ H_I^s(M)) \cong \mathrm{Ext}_R^s(R/I,\ M).$$

In this paper, we get the corresponding result for the local cohomology module with respect to (I, J). In Section 3, we prove the (I, J)-cofiniteness of $H_{I,J}^{\dim M}(M)$, which is a generalization of Theorem 3 in [4].

2 The Cofiniteness of $H_{I,J}^s(M)$

First, we give a theorem which is a generalization of Theorem 2.1 in [6] and Theorem B(β) in [7]. It is also a main result of this paper.