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T^*-extension of Lie Supertriple Systems

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Abstract: In this article, we study the Lie supertriple system (LSTS) T over a field $\mathbb{K}$ admitting a nondegenerate invariant supersymmetric bilinear form (call such a T metrisable). We give the definition of T^*_ω-extension of an LSTS T, prove a necessary and sufficient condition for a metrised LSTS (T, ϕ) to be isometric to a T^*-extension of some LSTS, and determine when two T^*-extensions of an LSTS are "same", i.e., they are equivalent or isometrically equivalent.

Key words: pseudo-metrised Lie supertriple system, metrised Lie supertriple system, T^*-extension

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1 Introduction

A Lie (super)triple system over a field $\mathbb{K}$ is called pseudo-metrisable if it admits an invariant nondegenerate bilinear form, and if further, the bilinear form can be chosen to be (super)symmetric, then T is called metrisable. Recently, metrisable Lie (super)triple systems have attracted a lot of attention due to its applications in the areas of mathematics and physics (see, for example, [1–6]).

The method of T^*-extension of Lie algebras was first introduced by Bordemann[7] in 1997 and this method is an important method for studying algebraic structures. In our early paper, we investigated the T^*-extension of Lie triple systems (see [6]). This paper is devoted to transfer the T^*-extension method to Lie supertriple systems.

Throughout this paper, all Lie supertriple systems considered are assumed to be of finite dimension over a field $\mathbb{K}$.

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2 Lie Supertriple Systems

In this section, we first briefly sketch the notion of a (pseudo-)metrisable Lie supertriple system.

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbf{Z}_2$-graded space over $\mathbb{K}$, where $V_{\bar{0}}$ and $V_{\bar{1}}$ are called bosonic and fermionic space, respectively, in physics literature. We denote the degree by

$$\deg(x) = \begin{cases} 0, & \text{if } x \in V_{\bar{0}}; \\ 1, & \text{if } x \in V_{\bar{1}}. \end{cases}$$

and write $(-1)^{xy} := (-1)^{\deg(x)\deg(y)}$.

Any element considered in this article is always assumed to be homogeneous, i.e., either $x \in V_{\bar{0}}$ or $x \in V_{\bar{1}}$.

Notice that the associate algebra $\mathrm{End}V$ is a superalgebra $\mathrm{End}V = \mathrm{End}_{\bar{0}}V \oplus \mathrm{End}_{\bar{1}}V$,

$$\mathrm{End}_\alpha V = \{a \in \mathrm{End}V \mid \ aV_s \subseteq V_{s+\alpha},\ s = \bar{0}, \bar{1}\}, \qquad \alpha = \bar{0}, \bar{1}.$$

Definition 2.1 *A Lie supertriple system (LSTS) is a $\mathbf{Z}_2$-graded space $T = T_{\bar{0}} \oplus T_{\bar{1}}$ over $\mathbb{K}$ with a trilinear composition $[\cdot, \cdot, \cdot]$, satisfying the following conditions*:

(1) $\deg([xyz]) = (\deg(x) + \deg(y) + \deg(z))(\mathrm{mod}\ 2)$;

(2) $[yxz] = -(-1)^{xy}[xyz]$;

(3) $(-1)^{xz}[xyz] + (-1)^{yx}[yzx] + (-1)^{zy}[zxy] = 0$;

(4) $[uv[xyz]] = [[uvx]yz] + (-1)^{(u+v)x}[x[uvy]z] + (-1)^{(u+v)(x+y)}[xy[uvz]]$.

An ideal of an LSTS T is a graded subspace I for which $[I, T, T] \subseteq I$. Moreover, if $[TII] = 0$, then I is called an abelian ideal of T. T is called abelian if it is an abelian ideal of itself. For any graded subspace V in T, the centralizer $Z_T(V)$ of V in T is defined by

$$Z_T(V) = \{x \in T \mid [xvt] = [xtv] = 0, \text{ for all } t \in T,\ v \in V\}.$$

In particular, $Z_T(T)$ is called the center of T and denoted simply by $Z(T)$. If T is an LSTS, define the lower central series for T by $T^0 := T$ and $T^{n+1} := [T^nTT]$ for $n \geq 0$. T is called nilpotent (of nilindex m) if there is a (smallest) positive integer m such that $T^m = 0$. Put $T^{(0)} := T$ and $T^{(n+1)} := [T^{(n)}TT^{(n)}]$. Then T is called solvable (of length k) if there is a (smallest) positive integer k such that $T^{(k)} = 0$.

Definition 2.2 *If an LSTS T admits a nondegenerate bilinear form b satisfying conditions*

(1) $b(x, y) = 0$ *unless* $d(x) = d(y)$; (*consistence*)

(2) $b([x, y, u],\ v) = -(-1)^{(x+y)u}b(u,\ [x, y, v])$, (*invariance*)

then we call T pseudo-metrisable and the pair (T, b) a pseudo-metrised LSTS. If, in addition, b satisfies also;

(3) $b(x, y) = (-1)^{xy}b(y, x)$, (*supersymmetry*)

then we call T metrisable and the pair (T, b) a metrised LSTS.

Proposition 2.1[1] *The following conditions are equivalent*:

(1) $b([x, y, u],\ v) = -(-1)^{(x+y)u}b(u,\ [x, y, v])$;