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On Results the Growth of Meromorphic Solutions of Algebraic Differential Equations

Su Xian-feng[1], Li Xiao-meng[1] and He Zhong-wei[2]

(*1. Department of Mathematics, Huaibei Normal University, Huaibei, Anhui, 235000*)

(*2. Department of Public Education, Jiangxi Bluesky University, Nanchang, 330029*)

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Abstract: In this paper, we give an estimate result of Gol'dberg's theorem concerning the growth of meromorphic solutions of algebraic differential equations by using Zalcman Lemma. It is an extending result of the corresponding theorem by Yuan *et al.* (Yuan W J, Xiao B, Zhang J J. The general theorem of Gol'dberg concerning the growth of meromorphic solutions of algebraic differential equations. *Comput. Math. Appl.*, 2009, **58**: 1788–1791). Meanwhile, we also take some examples to show that our estimate is sharp.

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1 Introduction and Main Results

We assume that the reader is familiar with the elementary Nevanlinna theory of meromorphic functions (see [1–3]). Meromorphic functions are always non-constant, if not otherwise specified.

In the past half century, many authors have studied the growth of meromorphic solutions of complex algebraic differential equations or the systems of complex algebraic differential equations in [4–7]. Recently, the main method for investigating the related problems was basically adapted from [8–9], which is called the Zalcman Lemma.

In order to state these results, we introduce some notations: $n \in \mathbf{N}^+ = \{1, 2, 3, \cdots\}$, $r_j \in \mathbf{N}$ for $j = 1, 2, \cdots, n$, and put $r = \{r_1, r_2, \cdots, r_n\}$. Define $\Omega(w)$ by

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E-mail address: suxianfeng2006@tom.com (Su X F).

$$\Omega[w](z) = \sum_{r \in I} a_r(z, w)\Omega_r[w](z), \tag{1.1}$$

where

$$\Omega_r[w](z) := [w']^{r_1}[w'']^{r_2} \cdots [w^{(n)}]^{r_n},$$

$a_r(z, w)$ is a rational function in both variables and I is a finite index set, and set $\Omega_0[w] = 1$. We call

$$u(r) = r_1 + 2r_2 + \cdots + nr_n$$

the weight of $\Omega_r(w)$ and

$$u = \max\{u_r \mid r \in I\}$$

the weight of $\Omega(w)$. $\deg_{z,\infty} a_r$ denotes the degree at infinity in variable z concerning $a_r(z, w)$, and

$$\deg_{z,\infty} a := \max\{\deg_{z,\infty} a_r,\ 0 \mid r \in I\}.$$

Let $\mathcal{F}$ be a family of meromorphic functions defined on a complex domain $\mathcal{D}$. $\mathcal{F}$ is said to be normal on $\mathcal{D}$, if for every sequence $\{f_n\} \in \mathcal{F}$, there exists a subsequence $\{f_{n_j}\}$ such that $\{f_{n_j}\}$ locally uniformly converges by spherical distance to a function $f(z)$ meromorphic in $\mathcal{D}$. Conversely, $\mathcal{F}$ is not normal on $\mathcal{D}$.

We define spherical derivative of the meromorphic function $w(z)$ by

$$w^\sharp(z) = \frac{|w'(z)|}{1 + |w(z)|^2}.$$

Bergweiler[5] considered the growth of the solutions of complex differential equation

$$(w')^n = \Omega[w], \tag{1.2}$$

where $\Omega[w]$ is a differential polynomial with the form (1.1), $a_r(z, w)$ is a rational function in z and w, and I is a finite index set.

As we all know, a research of the growth of meromorphic solution $w(z)$ of the differential equation (1.2) in the complex plane $\mathbf{C}$ has become one of important topics.

Bergweiler proved the following result.

Theorem 1.1[5] *Let $w(z)$ be any meromorphic solution of the algebraic differential equation (1.2), $n > u$. Then the growth order $\sigma(w)$ of $w(z)$ is finite.*

Yuan *et al.*[7] established a general estimate of growth order of $w(z)$, and obtained the following result.

Theorem 1.2[7] *Let $w(z)$ be meromorphic in the complex plane, $n \in \mathbf{N}^+$, $\Omega[w]$ be a differential polynomial with the form (1.1), and $n > u$. If $w(z)$ satisfies the differential equation (1.2), then the growth order $\sigma(w)$ of $w(z)$ satisfies*

$$\sigma(w) \le 2 + \frac{2\deg_{z,\infty} a}{n - u}.$$

Question 1.1 What is the result when the first-order derivative is replaced by kth-order in the left hand side of the equality (1.2) ($k \in \mathbf{N}^+$)?

In this paper, we give a general estimate of the order of $w(z)$, which depend on the degrees of coefficients of differential polynomial for $w(z)$, and it may be stated as follows.