COMMUNICATIONS IN MATHEMATICAL RESEARCH
29(4)(2013), 370–376

Branched Coverings and Embedded Surfaces in Four-manifolds

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Communicated by Lei Feng-chun

Abstract: In this paper, we get a lower bound on the genera of surfaces representing certain divisible classes in some 4-manifold X with $H_1(X;Z)$ finite.

Key words: embedded surface, 4-manifold, genus
2000 MR subject classification: 57R40, 57R95, 57N13
Document code: A
Article ID: 1674-5647(2013)04-0370-07

1 Introduction

One of the outstanding problems in four-dimensional topology is to find the minimal genus of an oriented smoothly embedded surface representing a given homology class in a smooth 4-manifold. There are a lot of results on this question in the literature. They can be divided into two classes, those proved by classical topological methods and those proved by methods of gauge theory. On the classical side, a major step forward was made by Rokhlin[1], Hsiang and Szczarba[2] who introduced branched covers to study this problem for divisible homology classes.

The applications of gauge theory to this problem were initially concerned with the representability of homology classes by spheres (see the Lawson's survey article [3]). They usually relied on Donaldson's early theorem (see [4]) on four-manifolds with $b_2^+ \leq 2$. Substantial progress was made recently by Kronheimer and Mrowka[5–6] who developed gauge theory for singular connection with non-trivial holonomy around a loop linking an embedded surface Σ. They used this to prove that if X has non-trivial Donaldson invariants (see [7]), then the genus $g(\Sigma)$ of Σ satisfies $g(\Sigma) \geq 1 + \dfrac{1}{2}\Sigma^2$ except that Σ is an inessential sphere or a sphere of self-intersection -1. Later Kotschick and Matić[8] combined the classical method using branched covers with gauge theoretic arguments to prove some results about the lower bound on the genera of surfaces representing certain divisible classes in 4-manifolds.

Received date: Oct. 2, 2011.
Foundation item: The NSF (10931005) of China.
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These results were obtained under the assumption that the 4-manifold is simply connected. In this paper, we discuss this problem for 4-manifold X with $H_1(X;\ Z)$ finite. Here we do not use Donaldson theory, but use some results obtained by using Seiberg-Witten theory. Together with the classical method for branched coverings, we get some results about the lower bound on the genera of surfaces representing certain divisible classes in some 4-manifold X with $H_1(X;\ Z)$ finite.

2 Branched Coverings and Preliminary Results

Let X be a closed 4-manifold with $H_1(X;\ Z)$ finite, $B \subset X$ be a connected closed oriented surface representing a homology class $v \in H_2(X;\ Z)$, and g be the genus of B. Split

$$H_2(X;\ Z) = \mathrm{Fr}H_2(X;\ Z) \oplus \mathrm{Tor}H_2(X;\ Z),$$

and denote by b_2 the second Betti number of X. Suppose that p is a prime such that p^r devides $v|_{\mathrm{Fr}H_2(X;\ Z)}$. Let $T \subset X$ be a tubular neighborhood of the branch locus B, and $W = \mathrm{closur}(X - T)$.

Lemma 2.1 *$H_1(W)$ is a finite group, and it contains a subgroup $G \cong Z_{p^r}$.*

Proof. By excision and duality, we have

$$H_1(W) \cong H^3(W,\ \partial W) \cong H^3(X,\ N) \cong H^3(X,\ B).$$

The exact sequence of the pair $(X,\ B)$ gives

$$\cdots \longrightarrow H^2(X) \xrightarrow{i^*} H^2(B) \xrightarrow{\sigma^*} H^3(X,\ B) \xrightarrow{j^*} H^3(X) \longrightarrow 0,$$

and since $H^2(B) \cong Z$, i^* is determined by

$$i^* |_{\mathrm{Fr}H^2(X)}\colon \mathrm{Fr}H^2(X) \to H^2(B).$$

Let

$$v|_{\mathrm{Fr}H_2(X)} = p^r u,$$

and choose a base $v_1,\ \cdots,\ v_{b_2}$ for $\mathrm{Fr}H_2(X)$ with $v_1 = u$. Let $\phi_1,\ \cdots,\ \phi_{b_2}$ be the dual base for $\mathrm{Fr}H^2(X)$, and $\mu \in H^2(B)$ dual to $[B] \in H_2(B)$. Since $i^*|_{\mathrm{Fr}H^2(X)} = \mathrm{Hom}(i^*, 1)$, we have

$$i^*(\phi_1) = p^r\mu, \qquad i^*(\phi_i) = 0, \quad i = 2, \cdots, b_2.$$

Let G be the subgroup of $H^3(X,\ B)$ generated by $\sigma^*\mu$. Then $G \cong Z_{p^r}$ and $G \subset \ker j^*$. Since $H^3(X) \cong H_1(X)$ is a finite group, $H^3(X,\ B)$ is a finite group. This completes the proof of Lemma 2.1.

Now let $V \to W$ be the unramified covering corresponding to the homomorphism

$$\pi_1(W) \to H_1(W) \to Z_{p^r}.$$

Since the normal bundle N_B has degree $[B]^2$, and $p^r \mid [B]^2$, there is a p^r-fold cover of N_B by the line bundle of degree $[B]^2/p^r$ given by $(w,\ x) \mapsto (w^{p^r},\ x)$. Using the identification of T with N_B we get a p^r cover $\tilde{T} \to T$ ramified along B. It is easy to see that we can constructed a p^r-fold branched covering $Y \to X$ with $Y = V \cup \tilde{T}$, branched along $B \subset X$.

Proposition 2.1 *If p^r is the maximal power of p dividing $v\ |_{\mathrm{Fr}H_2(X,\ Z)}$, then*

$$H_1(Y;\ Z_p) = 0.$$