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The Centres of Gravity of Periodic Orbits

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Abstract: Let $f : I \to I$ be a continuous map. If $P(n,\ f) = \{x \in I;\ f^n(x) = x\}$ is a finite set for each $n \in \mathbf{N}$, then there exits an anticentered map topologically conjugate to f, which partially answers a question of Kolyada and Snoha. Specially, there exits an anticentered map topologically conjugate to the standard tent map.

Key words: periodic orbit, centre of gravity, anticentered map, tent map

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1 Introduction

Let $(X,\ \rho)$ be a compact metric space and $f : X \to X$ a continuous map. $(X,\ f)$ is called a discrete dynamical system. The main task is to investigate how the points of X move, i.e., to understand the "orbits". For $x \in X$, the orbit of x under f is

$$\mathrm{Orb}(f,\ x) = \{x,\ f(x),\ f^2(x),\ \cdots\},$$

where $f^n = f \circ f \circ \cdots \circ f$ is the nth iteration of f obtained by composing f with itself n times. Periodic orbit is a simple type of orbit. If there exists a positive integer n such that $f^n(x) = x$, then $x \in X$ is called a periodic point and n is called a period of x. Furthermore, denote

$$P(n,\ f) = \{x \in I;\ f^n(x) = x\}.$$

Let $I = [0,\ 1]$. The dynamical systems on I have been well studied (see [1–2]). With regards to periodic points, there are also many results such as Sarkovskii's theorem (see [3–4]). In this article, we are interested in some properties of periodic orbits in dynamical systems $(I,\ f)$.

For a periodic point x of period p in a dynamical system $(I,\ f)$,

$$\gamma_f(x) = \frac{1}{p}\sum_{i=0}^{p-1} f^i(x)$$

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is said to be the centre of gravity of the orbit of x. A map $f : I \to I$ is said to be centered if the map $x \to \gamma_f(x)$ is a constant on the set of periodic points, i.e., each periodic orbit of f has the same centre of gravity; and f is said to be anticentered if any two different periodic orbits of f must have different centre of gravity.

Recall that a map $F : X \times I \to X \times I$ is called triangular (or skew product) if it has the form of

$$F(x,\ y) = (f(x),\ g(x,\ y)).$$

In the research of these maps, ones are interested in the standard tent map $\tau : I \to I$,

$$\tau(x) = 1 - |2x - 1|.$$

S. Kolyada and L. Snoha asked whether the standard tent map is anticentered, which was motivated by some results from [5]. M. Misiurewicz investigated that it is not anticentered since 22/127 and 26/127 belong to different periodic orbits of period 7 with the same centre of gravity 72/127 (see [6]). Furthermore, they asked the following question in [6].

Question A Let $f : I \to I$ be a continuous map. Can one always topologically conjugate f to an anticentered map $g : I \to I$?

We give a positive answer to Question A for the standard tent map. In fact, it holds for a special class of interval self-maps which contains the standard tent map.

Theorem 1.1 *Let $f : I \to I$ be a continuous map. If $P(n,\ f) = \{x \in I;\ f^n(x) = x\}$ is a finite set for each $n \in \mathbf{N}$, then there exists an anticentered map $g : I \to I$ topologically conjugate to f.*

Corollary 1.1 *Let τ be the standard tent map. Then there exists an anticentered map $g : I \to I$ topologically conjugate to τ.*

Proof. Since $P(n, \tau)$ is a finite set for each $n \in \mathbf{N}$, it is easy to get this conclusion by Theorem 1.1.

2 Preliminaries

In this section, we make some necessary preparation for proving Theorem 1.1. First of all, let us review some definitions and conclusions in topology (see [7]).

Let X and Y be two metric spaces. $C(X, Y)$ denotes the set of all continuous functions from X to Y. The metric d_Y on Y introduces a metric d on $C(X,\ Y)$ for all $f, g \in C(X, Y)$:

$$d(f,\ g) = \sup\{d_Y(f(x),\ g(x));\ x \in X\}.$$

Denote

$$S(X,\ Y) = \{f \in C(X,\ Y);\ f \text{ is surjective}\}$$

and by $\mathcal{H}(X)$ the autohomeomorphism group of X.