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On the Nonlinear Matrix Equation $\boldsymbol{X} + \boldsymbol{A}^* f_1(\boldsymbol{X})\boldsymbol{A} + \boldsymbol{B}^* f_2(\boldsymbol{X})\boldsymbol{B} = \boldsymbol{Q}$

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Abstract: In this paper, nonlinear matrix equations of the form $\boldsymbol{X} + \boldsymbol{A}^* f_1(\boldsymbol{X})\boldsymbol{A} + \boldsymbol{B}^* f_2(\boldsymbol{X})\boldsymbol{B} = \boldsymbol{Q}$ are discussed. Some necessary and sufficient conditions for the existence of solutions for this equation are derived. It is shown that under some conditions this equation has a unique solution, and an iterative method is proposed to obtain this unique solution. Finally, a numerical example is given to identify the efficiency of the results obtained.

Key words: nonlinear matrix equation, positive definite solution, iterative method

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1 Introduction

Denote by $P(n)$ the set of all $n \times n$ positive semi-definite matrices. In this paper, we consider the class of nonlinear matrix equations

$$\boldsymbol{X} + \boldsymbol{A}^* f_1(\boldsymbol{X})\boldsymbol{A} + \boldsymbol{B}^* f_2(\boldsymbol{X})\boldsymbol{B} = \boldsymbol{Q}, \tag{1.1}$$

where $\boldsymbol{A}$ is a nonsingular matrix, $\boldsymbol{Q}$ is a Hermitian positive definite matrix, f_1, f_2 are continuous maps from $P(n)$ into $P(n)$, and they are either monotone (meaning that $0 \leq \boldsymbol{X} \leq \boldsymbol{Y}$ implies that $f(\boldsymbol{X}) \leq f(\boldsymbol{Y})$) or anti-monotone (meaning that $0 \leq \boldsymbol{X} \leq \boldsymbol{Y}$ implies that $f(\boldsymbol{X}) \geq f(\boldsymbol{Y})$).

Nonlinear matrix equations of the form (1.1) often arise in dynamic programming, control theory, stochastic filtering, statistics, and so on. In recent years, these equations have been extensively studied by several authors, and some properties of their solutions have been

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obtained.

(1.1) has been investigated in some special cases. Some authors considered (1.1) in the case that $f_2(\boldsymbol{X}) = 0$ (see [1–2]). Du[3] treated the case that $f_1(\boldsymbol{X}) = -\boldsymbol{X}^{-\alpha}$, $f_2(\boldsymbol{X}) = -\boldsymbol{X}^{-\beta}$ and $\boldsymbol{Q} = \boldsymbol{I}$, where $\alpha, \beta \in (0, 1]$. For the case $f_2(\boldsymbol{X}) = 0$, many authors discussed the equation for particular choices of $f_1(\boldsymbol{X})$ and matrix $\boldsymbol{Q}$. For example, the case $f_1(\boldsymbol{X}) = \boldsymbol{X}^{-1}$ is studied in [4–5], the case $f_1(\boldsymbol{X}) = \boldsymbol{X}^{-2}$ and $\boldsymbol{Q} = \boldsymbol{I}$ is discussed in [6], and the case $f_1(\boldsymbol{X}) = \boldsymbol{X}^{-n}$ is considered in [7–8]. A more general case is $f_1(\boldsymbol{X}) = \boldsymbol{X}^{-q}$ where q is a positive number, which is discussed in [9–13]. Some other authors studied similar equations as in [14–16].

In this paper, we derive some necessary and sufficient conditions for the existence of the solutions of (1.1). And we investigate the uniqueness of the solution, and then we propose an iterative method to obtain this unique solution. Finally, we give a numerical example to identify the efficiency of the results obtained.

The following notations are used throughout this paper. For a positive definite matrix $\boldsymbol{A}$, $\lambda_{\max}(\boldsymbol{A})$ and $\lambda_{\min}(\boldsymbol{A})$ stand for the maximal and minimal eigenvalues of $\boldsymbol{A}$, respectively. $\boldsymbol{A}^*$ is the conjugate transpose of the matrix $\boldsymbol{A}$, and $\boldsymbol{A}^{-*}$ is the inversion of $\boldsymbol{A}^*$. $\|\boldsymbol{A}\|$ denotes the spectral norm of $\boldsymbol{A}$. $\boldsymbol{A} > 0$ ($\boldsymbol{A} \geq 0$) denotes that $\boldsymbol{A}$ is a positive definite (semi-definite) matrix, and $\boldsymbol{A} > \boldsymbol{B}$ ($\boldsymbol{A} \geq \boldsymbol{B}$) means $\boldsymbol{A} - \boldsymbol{B} > 0$ ($\boldsymbol{A} - \boldsymbol{B} \geq 0$). The notation $L_{\boldsymbol{A},\boldsymbol{B}}$ denotes the line segment joining $\boldsymbol{A}$ and $\boldsymbol{B}$, i.e.,

$$L_{\boldsymbol{A},\boldsymbol{B}} = \{t\boldsymbol{A} + (1-t)\boldsymbol{B} \mid t \in [0, 1]\}.$$

2 On the Positive Definite Solutions of (1.1)

Theorem 2.1 *If $f_i(\boldsymbol{D}^2) = (f_i(\boldsymbol{D}))^* f_i(\boldsymbol{D})$, $i = 1, 2$, for any nonsingular matrix $\boldsymbol{D}$, then* (1.1) *has an Hermite positive definite solution if and only if there is a nonsingular matrix $\boldsymbol{W}$ such that*

$$\boldsymbol{A} = [f_1((\boldsymbol{W}^*\boldsymbol{W})^{\frac{1}{2}})]^{-1}\boldsymbol{Z}_1\boldsymbol{Q}^{\frac{1}{2}}, \qquad \boldsymbol{B} = [f_2((\boldsymbol{W}^*\boldsymbol{W})^{\frac{1}{2}})]^{-1}\boldsymbol{Z}_2\boldsymbol{Q}^{\frac{1}{2}},$$

where $\boldsymbol{Q}^{-\frac{1}{2}}\boldsymbol{W}^*\boldsymbol{W}\boldsymbol{Q}^{-\frac{1}{2}} + \boldsymbol{Z}_1^*\boldsymbol{Z}_1 + \boldsymbol{Z}_2^*\boldsymbol{Z}_2 = \boldsymbol{I}$. In this case, (1.1) *has an Hermite positive definite solution $\boldsymbol{X} = \boldsymbol{W}^*\boldsymbol{W}$.*

Proof. If $\boldsymbol{X}$ is an Hermite positive definite solution of (1.1), then there is a unique Hermite positive definite matrix $\boldsymbol{W}$ such that $\boldsymbol{X} = \boldsymbol{W}^2$ (see [17–18]). Substituting $\boldsymbol{X} = \boldsymbol{W}^2$ into (1.1) gives

$$\boldsymbol{W}^2 + \boldsymbol{A}^* f_1(\boldsymbol{W}^2)\boldsymbol{A} + \boldsymbol{B}^* f_2(\boldsymbol{W}^2)\boldsymbol{B} = \boldsymbol{Q}.$$

Then we have

$$\boldsymbol{W}^*\boldsymbol{W} + \boldsymbol{A}^*(f_1(\boldsymbol{W}))^* f_1(\boldsymbol{W})\boldsymbol{A} + \boldsymbol{B}^*(f_2(\boldsymbol{W}))^* f_2(\boldsymbol{W})\boldsymbol{B} = \boldsymbol{Q}.$$

$\boldsymbol{Q}$ is Hermite positive definite, so

$$\begin{aligned}&(\boldsymbol{Q}^{-\frac{1}{2}})^*\boldsymbol{W}^*\boldsymbol{W}\boldsymbol{Q}^{-\frac{1}{2}} + (\boldsymbol{Q}^{-\frac{1}{2}})^*\boldsymbol{A}^*(f_1(\boldsymbol{W}))^* f_1(\boldsymbol{W})\boldsymbol{A}\boldsymbol{Q}^{-\frac{1}{2}}\\ &\quad + (\boldsymbol{Q}^{-\frac{1}{2}})^*\boldsymbol{B}^*(f_2(\boldsymbol{W}))^* f_2(\boldsymbol{W})\boldsymbol{B}\boldsymbol{Q}^{-\frac{1}{2}} = \boldsymbol{I}.\end{aligned} \tag{2.1}$$

Let $\boldsymbol{Z}_1 = f_1(\boldsymbol{W})\boldsymbol{A}\boldsymbol{Q}^{-\frac{1}{2}}$, $\boldsymbol{Z}_2 = f_2(\boldsymbol{W})\boldsymbol{B}\boldsymbol{Q}^{-\frac{1}{2}}$. Then

$$\boldsymbol{A} = (f_1(\boldsymbol{W}))^{-1}\boldsymbol{Z}_1\boldsymbol{Q}^{\frac{1}{2}} = (f_1((\boldsymbol{W}^*\boldsymbol{W})^{\frac{1}{2}}))^{-1}\boldsymbol{Z}_1\boldsymbol{Q}^{\frac{1}{2}}, \qquad \boldsymbol{B} = (f_2((\boldsymbol{W}^*\boldsymbol{W})^{\frac{1}{2}}))^{-1}\boldsymbol{Z}_2\boldsymbol{Q}^{\frac{1}{2}}.$$