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Multi-point Boundary Value Problems for Nonlinear Fourth-order Differential Equations with All Order Derivatives

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Abstract: By using fixed point theorem, multiple positive solutions for some fourth-order multi-point boundary value problems with nonlinearity depending on all order derivatives are obtained. The associated Green's functions are also given.
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1 Introduction

In this paper, we are interested in the positive solution for fourth-order nonlinear differential equation

$$x^{(4)}(t) = f(t,\ x(t),\ x'(t),\ x''(t),\ x'''(t)), \qquad t \in [0, 1], \tag{1.1}$$

subject to the boundary conditions

$$x'''(1) = 0, \quad x''(0) = 0, \quad x'(0) = 0, \quad x(1) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \tag{1.2}$$

or

$$x'''(1) = 0, \quad x''(0) = 0, \quad x'(1) = 0, \quad x(0) = \sum_{i=1}^{m-2} \beta_i x(\xi_i), \tag{1.3}$$

where $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$, $0 < \beta_i < 1$, $i = 1, 2, \cdots, m-2$, $\sum_{i=1}^{m-2} \beta_i < 1$ and $f \in C([0, 1] \times \mathbf{R}^4, [0, +\infty))$.

It is well known that the boundary value problems of nonlinear differential equations arise in a large number of problems in physics, biology and chemistry. For example, the

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deformations of an elastic beam in the equilibrium state can be described as a boundary value problem of some fourth-order differential equations. Owing to its importance in application, the existence of positive solutions for nonlinear second-order or high-order boundary value problems have been studied by many authors (see [1–15]).

When it comes to positive solutions of nonlinear fourth-order boundary value problems, the different two point boundary value problems are considered by many authors (see [16–24]). Few paper deals with the multi-point cases. Furthermore, for nonlinear fourth-order equations, many results were established under the case that the nonlinear term does not depend on the first, second and third order derivatives in [16–23]. Few paper deals with the positive solutions under the situation that all order derivatives are involved in the nonlinear term explicitly (see [25–27]). In fact, the derivatives are of great importance in the problem in some cases. For example, this is the case in the linear elastic beam equation (Euler-Bernoulli equation)

$$(EIu''(t))'' = f(t), \qquad t \in (0, L),$$

where $u(t)$ is the deformation function, L is the length of the beam, $f(t)$ is the load density, E is the Young's modulus of elasticity and I is the moment of inertia of the cross-section of the beam. In this problem, the physical meaning of the derivatives of the function $u(t)$ is as follows: $u^{(4)}(t)$ is the load density stiffness, $u'''(t)$ is the shear force stiffness, $u''(t)$ is the bending moment stiffness and the $u'(t)$ is the slope (see [24]). If the payload depends on the shear force stiffness, bending moment stiffness or the slope, the derivatives of the unknown function are involved in the nonlinear term explicitly.

The goal of the present paper is to study the fourth-order multi-point boundary value problems (1.1)-(1.2) and (1.1)-(1.3), in which all order derivatives are involved in the nonlinear term explicitly. In this sense, the problem studied in this paper are more general than before. In order to overcome the difficulty of the derivatives that appear, our main technique is to transfer the problem into an equivalent operator equation by constructing the associate Green's function and apply a fixed point theorem due to [28]. In this paper, multiple monotone positive solutions for the problems (1.1)-(1.2) and (1.1)-(1.3) are established. The results extend the study for fourth-order boundary value problems of nonlinear ordinary differential equations.

2 Preliminaries and Lemmas

In this section, some preliminaries and lemmas used later are presented.

Definition 2.1 *The map α is said to be a nonnegative continuous convex functional on a cone P of a real Banach space E provided that $\alpha : P \to [0, +\infty)$ is continuous and*

$$\alpha(tx + (1-t)y) \le t\alpha(x) + (1-t)\alpha(y), \qquad x, y \in P, \ t \in [0, 1].$$

Definition 2.2 *The map β is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E provided that $\beta : P \to [0, +\infty)$ is continuous and*

$$\beta(tx + (1-t)y) \ge t\beta(x) + (1-t)\beta(y), \qquad x, y \in P, \ t \in [0, 1].$$