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The Influence of Primitive Subgroups on the Structure of Finite Groups

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Abstract: A subgroup H of a group G is said to be primitive if it is a proper subgroup of the intersection of all subgroups of G containing H as its proper subgroup. The purpose of this note is to go further into the influence of primitive subgroups on the structure of finite groups. Some new results are obtained.

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1 Introduction

All groups considered in this paper are finite and G denotes a finite group. The generalized concept of maximal subgroups of a group G, namely, the primitive subgroup, was introduced by Johnson[1] in 1971. He called a subgroup H of a group G primitive if it is a proper subgroup of the intersection of all subgroups of G containing H as its proper subgroup. We denote by $H <_{\text{prim}} G$ that H is a primitive subgroup of G. It is interesting to note that every group G has a primitive subgroup and that every proper subgroup of G is the intersection of some primitive subgroups of G. Since the intersection of all primitive subgroups is the identity subgroup, we can easily see that the class of all primitive subgroups is obviously wider than the class of all maximal subgroups. Guo, Shum and Skiba[2] gave the structure of the finite group in which every primitive subgroup has a prime power index. They proved that every primitive subgroup of a finite group G has a prime-power index if and only if $G = [D]M$ is a supersoluble group, where D and M are nilpotent Hall subgroups of G, D is the smallest term of the lower central series of G and $G = DN_G(D \cap X)$ for every primitive subgroup X of G.

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2 Preliminaries

Recall that the quaternion group is a 2-group with a unique element of order 2, and the generalized quaternion group Q_{2^n} of order 2^n is the group with the following presentation of the form:

$$Q_{2^n} = \langle a,\ b \mid a^{2^{2-1}} = 1,\ b^2 = a^{2^{n-2}},\ a^b = a^{-1},\ n \geq 3\rangle.$$

Note that a group G is said to be primary if the order of G is a prime power.

Lemma 2.1 ([3], Theorem III.8.2) *If a p-group has a unique subgroup of order p, then G is a cyclic group or a generalized quaternion group.*

Lemma 2.2 [1] *If every primitive subgroup of G has a prime power index, then G is supersoluble.*

3 Main Results

Proposition 3.1 *If $H <_{\text{prim}} G$, then $N_G(H)/H$ is either a cyclic p-group for some prime p or a generalized quaternion group.*

Proof. Suppose that the factor group $N_G(H)/H$ had two different subgroups A/H and B/H of prime order. Then H were a proper subgroup of A and of B. H were a primitive subgroup of G, and

$$H < K = A \cap B.$$

This implies that

$$K = A = B.$$

The contradiction shows that $N_G(H)/H$ has a unique subgroup of prime order. By Sylow theorem, $N_G(H)/H$ is a primary group. Thus, by Lemma 2.1, we obtain that $N_G(H)/H$ is either a cyclic p-group for some prime p or a generalized quaternion group.

Proposition 3.2 *If the identity subgroup is primitive in G, then G is either a cyclic p-group for some prime p or a generalized quaternion group.*

Proof. Obviously, G is a p-group for some prime p. If G had two subgroups P_1 and P_2 of order p, then $1 < P_1$, $1 < P_2$ and $1 = P_1 \cap P_2$, which contradicts the fact that 1 is a primitive subgroup. Hence G has only a subgroup of order p. By Lemma 2.1, we see that G is either a cyclic p-group or a generalized quaternion group. This completes the proof.

As usual, a subgroup H of G is said to be non-trivial if H is neither an identity subgroup nor G.