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A Joint Density Function in the Renewal Risk Model*

Xu Huai[1] and Tang Ling[2]

(*1. School of Mathematics, Anhui University, Hefei, 230039*)

(*2. Department of mathematics, Anhui Institute of Architecture and Industry, Hefei, 230601*)

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Abstract: In this paper, we consider a general expression for $\phi(u,\ x,\ y)$, the joint density function of the surplus prior to ruin and the deficit at ruin when the initial surplus is u. In the renewal risk model, this density function is expressed in terms of the corresponding density function when the initial surplus is 0. In the compound Poisson risk process with phase-type claim size, we derive an explicit expression for $\phi(u,\ x,\ y)$. Finally, we give a numerical example to illustrate the application of these results.

Key words: deficit at ruin, surplus prior to ruin, phase-type distribution, renewal risk model, maximal aggregate loss

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1 Introduction

The renewal risk model $\{U(t)\}_{t\geq 0}$ is defined by

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i,$$

where u is the initial surplus, c is the rate of premium income per unit time, $\{X_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed (i.i.d.) random variables, where X_i represents the amount of the ith claim, and $\{N(t)\}_{t\geq 0}$ is a counting process with $N(t)$ denoting the number of claims up to time t. In addition, X_i has a density function $\theta(x)$ and a distribution function

$$\Theta(x) = 1 - \bar{\Theta}(x) = P\{X \leq x\},$$

where X is an arbitrary X_i. Let

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$$E(X) = \int_0^\infty x \mathrm{d}\Theta(x) < \infty.$$

The sequence of i.i.d. random variables $\{W_i\}_{i=1}^\infty$ represents the claim inter-arrival times, with W_1 being the time until the first claim. W_i has a density function $k(t)$ and a distribution function

$$K(t) = 1 - \bar{K}(t) = P\{W \le t\},$$

where W is an arbitrary W_i. Let

$$E(W) = \int_0^\infty t \mathrm{d}K(t) < \infty.$$

We assume that claim amounts are independent of claim inter-arrival times. Further, we assume that

$$cE(W) > E(X).$$

Define the time of ruin

$$T = \inf\{t : U(t) < 0\},$$

where $T = \infty$ if $U(t) \ge 0$ for all $t > 0$. Denote the ruin probability by

$$\psi(u) = P\{T < \infty \mid U(0) = u\},$$

and the survival probability by

$$\delta(u) = 1 - \psi(u).$$

It is well known that

$$\psi(u) = P\{L > u\} = \sum_{n=1}^\infty (1-\rho)\rho^n \bar{F}^{*n}(u), \qquad u \ge 0, \tag{1.1}$$

where $\rho = \psi(0)$, L is the well-known maximal aggregate loss in the renewal risk model, and

$$F(y) = 1 - \bar{F}(y)$$

is the so-called ladder height distribution function, which can be interpreted as either the distribution function of the deficit at ruin when initial surplus $u = 0$ or the distribution function of the amount of a drop in surplus, given that a drop below its initial level occurs.

$$F^{*n}(y) = 1 - \bar{F}^{*n}(y)$$

is the distribution function of the n-fold convolution of $F(y)$ with itself (see [1]).

Let

$$\begin{aligned}\varPhi(u,\ x,\ y) &= \int_0^x \int_0^y \phi(u,\ r,\ s)\mathrm{d}s\mathrm{d}r \\ &= P\{U(T_-) \le x,\ |U(T)| \le y,\ T < \infty \mid U(0) = u\},\end{aligned}$$

where $U(T_-)$ denotes the surplus prior to ruin, and $U(T)$ denotes the deficit at ruin. $\varPhi(u,\ x,\ y)$ may be interpreted as the probability that ruin occurs from initial surplus u with the deficit at ruin no greater than y and the surplus prior to ruin no greater than x. $\phi(u,\ r,\ s)$ denotes the joint density function. Let

$$h(u,\ x) = \int_0^\infty \phi(u,\ x,\ y)\mathrm{d}y,$$