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A Maschke Type Theorem for Doi-Hopf π-modules*

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Abstract: We prove a Maschke type theorem for Doi-Hopf π-modules. A sufficient condition for having a Maschke type property is that there exists a suitable total integral map for the Doi-Hopf π-modules in question. The applications of the results are considered. Finally, As an application of the existence of total integral, we prove that $\bigoplus_{\alpha\in\pi} C_\alpha \otimes A$ is a generator in the category ${}^{\pi\text{-}C}\mathcal{U}(H)_A$.

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1 Introduction

A clue result in classical representation theory is Maschke's theorem, which states that a group ring over a finite group is semisimple if and only if the characteristic of the field does not divide the order of the group. Keeping in mind the fact that a group ring is an example of a Hopf algebra, one can ask if Maschke's theorem can be generalized to finite-dimensional Hopf algebras. Larson and Sweedler[1] proved that a finite-dimensional Hopf algebra is semisimple if and only if there exists a left or right integral in H, such that the image of this integral under the augmentation map equals 1. If one works over a fixed field k, the semisimplicity is replaced by the following condition: every short exact sequence of H-modules that splits as a sequence of k-linear spaces, also splits as a sequence of H-modules.

Several other generalizations of Maschke's theorem have appeared in the literature. Doi[2] proved a Maschke type theorem for relative Hopf modules under some restrictive conditions which can be got rid of. Soon afterwards, a Maschke type theorem for Doi-Hopf modules was proved by Caenepeel *et al.*[3] Recently, Wang[4] generalized Maschke's theorem to Hopf

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π-comodules. A natural question occurs to us that whether Maschke's theorem can be generalized to Doi-Hopf π-modules.

The purpose of this paper is devoted to solving the question presented above, but we need to overcome the key question, i.e., how to introduce the notion of integral map for Doi-Hopf π-datum. In 2002, Menini and Militaru[5] defined the more general concept of an integral associated to a three tuple $(H,\ A,\ C)$ called the Doi-Koppinen datum, consisting of a Hopf algebra H which coacts on an algebra A and acts on a coalgebra C. Inspired by the idea in [5], we introduce the notion of integral map for Doi-Hopf π-datum (see Definition 3.1). In the sequel, we present Maschke's theorem for Doi-Hopf π-modules.

This paper is organized as follows. In Section 2, we recall definitions and basic results related to Hopf group-coalgebras which are needed later. In Section 3, the notions of integral are introduced, then we prove a Maschke type theorem for Doi-Hopf π-modules (see Theorem 3.1), which applies to Doi-Hopf modules (see Corollary 3.1) and Hopf π-comodules (see Corollary 3.2). Finally, we consider the application of the existence of the total integral (see Theorem 3.2).

2 Preliminaries

In this section, we recall definitions and discuss properties of Hopf group-coalgebras and comodule algebras. Most of the materials presented here can be found in [4, 6–12].

Throughout this paper, we always let π be a finite discrete group with a neutral element e and k be a field. If a tensor product is written without index, then it is assumed to be taken over k, that is, $\otimes = \otimes_k$. If U and V are k-spaces, then $T_{U,V} : U \otimes V \to V \otimes U$ denote the flip map defined by

$$T_{U,V}(u \otimes v) \to v \otimes u, \qquad u \in U,\ v \in V.$$

2.1 The π-coalgebras

A π-coalgebra is a family of k-spaces

$$C = \{C_\alpha\}_{\alpha\in\pi}$$

together with a family of k-linear maps

$$\Delta = \{\Delta_{\alpha,\beta} : C_{\alpha\beta} \to C_\alpha \otimes C_\beta\}_{\alpha,\beta\in\pi}$$

(called a comultiplication) and a k-linear map

$$\varepsilon : C_e \to k$$

(called a counit) such that Δ is coassociative in the sense that

$$(\Delta_{\alpha,\beta} \otimes id_{C\gamma}) \circ \Delta_{\alpha\beta,\gamma} = (id_{C_\alpha} \otimes \Delta_{\beta,\gamma}) \circ \Delta_{\alpha,\beta\gamma}, \qquad \alpha, \beta, \gamma \in \pi$$

and

$$(id_{C_\alpha} \otimes \varepsilon) \circ \Delta_{\alpha,e} = id_{C_\alpha} = (\varepsilon \otimes id_{C_\alpha}) \circ \Delta_{e,\alpha}, \qquad \alpha \in \pi.$$

Remark 2.1 $(C_e,\ \Delta_{e,e},\ \varepsilon)$ is an ordinary coalgebra in the sense of Sweedler. Following the Sweedler's notation for π-coalgebras, for any $\alpha, \beta \in \pi$ and $c \in C_{\alpha\beta}$, we write

$$\Delta_{\alpha,\beta}(c) = c_{(1,\alpha)} \otimes c_{(2,\beta)}.$$