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Derivation Algebra of Quasi R_n-filiform Lie Algebra*

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Abstract: In this paper we explicitly determine the derivation algebra of a quasi R_n-filiform Lie algebra and prove that a quasi R_n-filiform Lie algebra is a completable nilpotent Lie algebra.

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1 Introduction

The classification of Lie algebras is the most important task in Lie theory. There are two ways to get the classification of Lie algebras: by dimension or by structure. The dimension approach have got a lot of useful results and have some interesting applications in general relativity. However, it seems to be neither feasible, nor fruitful to proceed by dimension in the classification of Lie algebras when its dimension is beyond 6. We then turn to the structure approach. In 1966, Vergne[1] applied the cohomology theory of Lie algebra to study on variety of nilpotent Lie algebras. From then on, naturally graded filiform Lie algebras and its deformations, especially L_n filiform Lie algebras have been central research object for the last thirty years. R_n filiform Lie algebra is an important filiform Lie algebra which plays a role in the classification of rigid Lie algebras in [2]. Complete Lie algebras (i.e., centerless with only inner derivations) first appeared in 1951, in the context of Schenkman's theory of subinvariant Lie algebras in [3]. In recent years, different authors have concentrated on classifications and structural properties of complete Lie algebras in [4–9]. In this paper, we study a class of nilpotent algebras called quasi R_n-filiform Lie algebras. We explicitly determine the derivation algebra and prove that a quasi R_n-filiform Lie algebra is a completable nilpotent Lie algebra (i.e., the nilradical of a complete solvable Lie algebra).

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If H is a maximal torus on N (a maximal abelian subalgebra of Der$N$, which consists of semisimple linear transformations), then N can be decomposed into a direct sum of root spaces with respect to

$$H : N = \sum_{\alpha \in H^*} N_\alpha.$$

Definition 1.1[10] *Let H be a maximal torus on N. A minimal system of generators which consists of root vectors for H is called an H-*msg.

Lemma 1.1[11] *Let H_1 and H_2 be two maximal tori on N. Then there exists $\theta \in$*Aut*N, such that $H_2 = \theta H_1 \theta^{-1}$.*

As all maximal tori on N are mutually conjugated, the dimension of a maximal torus on N is an invariant of N called the rank of N (denoted by rank(N)).

If H is a maximal torus on the nilpotent Lie algebra N, and we define the bracket in $H \dot{+} N$ by $[h_1 + n_1, h_2 + n_2] = h_1(n_2) - h_2(n_1) + [n_1, n_2]$, where $h_i \in H$, $n_i \in N$, $i = 1, 2$, then $H \dot{+} N$ is a solvable Lie algebra.

Definition 1.2[9] *Let H be a maximal torus on the nilpotent Lie algebra N. If the solvable Lie algebra $H \dot{+} N$ is complete, then N is called a completable nilpotent Lie algebra.*

Theorem 1.1[5] *Let H be a maximal torus on N, $\{x_1, x_2, \cdots, x_n\}$ be an H-*msg*, where $x_i \in N_{\alpha_i}$, and there be no zero root space in the decomposition of $N = \sum\limits_{\alpha \in H^*} N_\alpha$ with respect to H. Then N is a completable nilpotent Lie algebra if and only if the following condition holds: for all $\varphi \in$ Der$N$, if $\varphi N_{\alpha_i} \subseteq N_{\alpha_i}$, then $\varphi x_i = k_i x_i$, where $k_i \in \mathbf{C}$.*

Definition 1.3[2] *A filiform Lie algebra R_n is an $(n+1)$-dimensional filiform Lie algebra defined on the basis $\{e_0, e_1, \cdots, e_n\}$ by $[e_0, e_i] = e_{i+1}$, $1 \le i \le n-1$, $[e_1, e_i] = e_{i+2}$, $2 \le i \le n-2$, the undefined brackets being zero or obtained by antisymmetry.*

Definition 1.4[12] *If $N = N_1 + N_2 + \cdots + N_m$, where $N_i \cong R_n$, $1 \le i \le m$, and*

$$[N_i, N_j] = 0, \qquad i \ne j,$$

then N is called a quasi R_n-filiform Lie algebra, and we denote N by $N(R_n, m, r)$, where

$$r = \dim C^{n-2} N.$$

Remark 1.1 The sum in the decomposition $N = N_1 + N_2 + \cdots + N_m$ is not necessarily direct, so the subalgebra in the decomposition can have a nontrivial intersection. Obviously, $N(R_n, m, r)$ also admits an R_n-filiform decomposition

$$N(R_n, m, r) = N_1(R_n, m_1, 1) + N_2(R_n, m_2, 1) + \cdots + N_q(R_n, m_q, 1),$$

where $N_i \cap N_j = [N_i, N_j] = 0$, $i \ne j$, $\sum\limits_{i=1}^{q} m_i = m$.

Lemma 1.2 *Let $\{e_{i0}, e_{i1}, \cdots, e_{in}\}$ be the basis of N_i, where N_i is as in the decomposition $N = N_1 + N_2 + \cdots + N_m$. Then $\{e_{i0}, e_{i1}, \cdots, e_{i,n-1} \mid 1 \le i \le m\}$ is linearly independent. Let $\{e_{i0}, e_{i1}, \cdots, e_{i,n-1}, e_{q_1,n}, e_{q_2,n}, \cdots, e_{q_k,n} \mid 1 \le i \le m\}$ be a basis of $N(R_n, m, r)$. Then $\{e_{q_1,n}, e_{q_2,n}, \cdots, e_{q_k,n}\}$ is a basis of $C^{n-2}N$.*