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A Penalty Approach for Generalized Nash Equilibrium Problem*

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Abstract: The generalized Nash equilibrium problem (GNEP) is a generalization of the standard Nash equilibrium problem (NEP), in which both the utility function and the strategy space of each player depend on the strategies chosen by all other players. This problem has been used to model various problems in applications. However, the convergent solution algorithms are extremely scare in the literature. In this paper, we present an incremental penalty method for the GNEP, and show that a solution of the GNEP can be found by solving a sequence of smooth NEPs. We then apply the semismooth Newton method with Armijo line search to solve latter problems and provide some results of numerical experiments to illustrate the proposed approach.

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1 Introduction

We consider the generalized Nash equilibrium problem. We first recall the definition of the Nash equilibrium problem. Let N be the number of players, and each player $\nu \in \{1, \cdots, N\}$ control a variable $\boldsymbol{x}^\nu \in \mathbf{R}^{n_\nu}$. All players' strategies are collectively denoted by a vector $\boldsymbol{x} = (\boldsymbol{x}^1, \cdots, \boldsymbol{x}^N)^T \in \mathbf{R}^n$, where $n = n_1 + \cdots + n_N$. To emphasize the ν-th player's variables within the vector $\boldsymbol{x}$, we sometimes write $\boldsymbol{x} = (\boldsymbol{x}^\nu,\ \boldsymbol{x}^{-\nu})^T$, where $\boldsymbol{x}^{-\nu} \in \mathbf{R}^{n_{-\nu}}$ subsumes all the other players' variables.

Let $\theta^\nu : \mathbf{R}^n \to \mathbf{R}$ be the ν-th player's payoff (or loss or utility) function, and $X^\nu \subseteq \mathbf{R}^{n_\nu}$ be the strategy set of the player ν. Then $\boldsymbol{x}^* = (\boldsymbol{x}^{*,1}, \cdots, \boldsymbol{x}^{*,N})^T \in \mathbf{R}^n$ is called a Nash equilibrium, or a solution of the standard Nash equilibrium problem (NEP), if each block

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component $\boldsymbol{x}^{*,\nu}$ is a solution of the optimization problem:

$$\begin{aligned}&\min_{\boldsymbol{x}^\nu} \theta^\nu(\boldsymbol{x}^\nu,\ \boldsymbol{x}^{*,-\nu})\\ &\text{s.t. } \boldsymbol{x}^\nu \in X^\nu.\end{aligned}$$

The generalized Nash equilibrium problem (GNEP) generalizes the situation to some extent since now the strategy of player ν belongs to a set $X_\nu(\boldsymbol{x}^{-\nu}) \subseteq \mathbf{R}^{n_\nu}$ that depends on the rival players' strategies. The aim of each player ν, giving the other players' strategies $\boldsymbol{x}^{-\nu}$, is to choose a strategy $\boldsymbol{x}^\nu$ that solves the minimization problem:

$$\begin{aligned}&\min_{\boldsymbol{x}^\nu} \theta^\nu(\boldsymbol{x}^\nu,\ \boldsymbol{x}^{-\nu})\\ &\text{s.t. } \boldsymbol{x}^\nu \in X_\nu(\boldsymbol{x}^{-\nu}).\end{aligned}$$

A vector $\boldsymbol{x} = (\boldsymbol{x}^\nu)_{\nu=1}^N$ is said to be feasible to the GNEP if $\boldsymbol{x}^\nu \in X_\nu(\boldsymbol{x}^{-\nu})$ for all ν. The GNEP is the problem to find a feasible point $\boldsymbol{x}^*$ such that each player's strategy $\boldsymbol{x}^{*,\nu}$ satisfies

$$\theta^\nu(\boldsymbol{x}^{*,\nu},\ \boldsymbol{x}^{*,-\nu}) \leq \theta^\nu(\boldsymbol{y}^\nu,\ \boldsymbol{x}^{*,-\nu}), \qquad \boldsymbol{y}^\nu \in X_\nu(\boldsymbol{x}^{*,-\nu}).$$

Such a vector $\boldsymbol{x}^*$ is called a generalized Nash equilibrium or a solution of the GNEP.

Since Arrow and Debreu's paper [1] on the existence of equilibria for a competitive economy was published, NEPs and GNEPs have been the subject of a constant if not intense interest. Especially in recent years, the GNEP has been used to model a host of interesting problems arising in economy, computer science, telecommunications and deregulated markets (for example, see [2–4]).

Numerical methods for the GNEP have been developed with different objectives and problem settings. Motivated by the fact that a standard NEP can be reformulated as a variational inequality problem (VI) (see [5–6]), Harker[7] also gave a reformulation of the GNEP as a quasi-variational inequality (QVI). It was noted in [8] that the normalized Nash equilibria which is a solution of the GNEP can be found by solving a suitable standard VI associated to the GNEP. This method seems to be promising because we can find a solution of the GNEP by solving a single VI. The obtained solution is a particular equilibrium. In general, a GNEP has multiple or even infinitely many solutions, and therefore, the VI reformulation fail to identify some important GNEP solutions.

The penalization approach to GNEPs was initiated very recently by Fukushima and Pang[9]. The idea of using an exact penalty approach by which a single nondifferentiable NEP has to be solved to obtain a solution of the GNEP was first put forward in [10], in a rather sketch way. This topic has recently been considered also by Fukushima[11], Facchinei and Kanzow[12]. They gave some conditions under which penalty approaches can be used to find a generalized Nash equilibrium.

In this paper, by using the logarithmic barrier function and quadratic penalty terms, we transform a class of GNEPs which have shared equality constraints into a sequence of NEPs and show that a solution of the GNEP can be found under suitable conditions. Next, we apply the semismooth Newton method to solve latter NEPs. Finally, we present some numerical results to illustrate the proposed approach.

We use the following notations throughout the paper. For a differentiable function g : $\mathbf{R}^n \to \mathbf{R}^m$, the Jacobian matrix of g at $\boldsymbol{x} \in \mathbf{R}^n$ is denoted by $\mathcal{J}g(\boldsymbol{x})$, and its transpose