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The Existence of Coupled Solutions for a Kind of Nonlinear Operator Equations in Partial Ordered Linear Topology Space*

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Abstract: The main purpose of this paper is to examine the existence of coupled solutions and coupled minimal-maximal solutions for a kind of nonlinear operator equations in partial ordered linear topology spaces by employing the semi-order method. Some new existence results are obtained.

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1 Introduction

The techniques of partial order theory are used to discuss the existence of coupled solutions and coupled minimal-maximal solutions for a kind of nonlinear operator equation in a partial ordered linear topology space as follows:

$$Nx = A(x, x), \tag{1.1}$$

where N is an increasing operator and A is a mixed monotone operator.

In 1987, Guo and Lakshmikantham[1] studied a nonlinear operator equation in a Banach space as

$$x = A(x, x), \tag{1.2}$$

where A is a mixed monotone operator. They obtained some existence results of coupled solution for this operator equation. In 2005, Liu and Feng[2] considered the following operator equation:

$$Nx = Ax \tag{1.3}$$

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in a complete metric space and a Banach space, respectively, and by using the technique of partial order theory they obtained some existence results of solution. Very recently, He[3] has dealt with the operator equation (1.1) in Banach spaces and have given some solvability results for this kind of equations by using the concept of ϕ concave-ψ convex operator (see [4]).

Motivated and inspired by the above works, the main purpose of this paper is to further study the solvability of the equation (1.1). Under some suitable conditions, we give some new existence theorems for this kind of equations. To the knowledge of the author, there are very few works on the existence of coupled solutions and coupled minimal-maximal solutions for the equation (1.1) in partial ordered linear topology space, and therefore, our results generalize and improve some corresponding results.

2 Preliminaries

In this section, we give some concepts and lemmas which are necessary for proving the main results of this paper, and the other unstated concepts can be seen in [5–8].

Let E be a real linear topology space, P be a cone of E and " $\leq$ " be a partial order induced by the cone P, i.e., for any x, $y \in E$, $x \leq y$ (or alternatively, denoted as $y \geq x$) if and only if $y - x \in P$. We write $x < y$, if $x \leq y$ and $x \neq y$.

Let x, $y \in E$, $x < y$. The set defined by $[x, y] = \{z \,|\, x \leq z \leq y\}$ is called an ordered interval in E. For any subset $D \subset E \times E$, we denote by $\bar{D}^w$, $\overline{\text{co}}(D)$ and CD the weak closure of D, the closed convex hull of D and the complement of D, respectively.

Let

$$P_1 = \{(x, y) \in E \times E \mid x \geq \theta,\ y \leq \theta\},$$

where θ denotes the zero element of E. It is easy to see that P_1 is a cone of the product space $E \times E$, and P_1 defines a partial order in $E \times E$ as follows (denoted as $\prec$):

$(x, y) \prec (u, v)$ (or alternatively, denoted as $(u, v) \succ (x, y)$) if and only if $x \leq u$ and $y \geq v$.

Definition 2.1[9−10] *Let D be a nonempty subset of a real partial order linear topology space $(E, \leq)$.*

(i) *The operator $A : D \times D \to E$ is said to be mixed monotone if $A(x, y)$ is both non-decreasing in x and nonincreasing in y, i.e., if $u_1 \leq u_2$, $v_2 \leq v_1$, u_i, $v_i \in D$ $(i = 1, 2)$ imply*

$$A(u_1, v_1) \leq A(u_2, v_2).$$

(ii) *A point $(x^*, y^*) \in D \times D$, $x^* \leq y^*$ is called a coupled solution of the nonlinear operator equation* (1.1) *if*

$$Nx^* = A(x^*, y^*), \qquad A(y^*, x^*) = Ny^*.$$

(iii) *A point $(x^*, y^*) \in D \times D$, $x^* \leq y^*$ is called a coupled minimal-maximal solution of the nonlinear operator equation* (1.1), *if (x^*, y^*) is a coupled solution of the nonlinear*