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Approximation of the Nearest Common Fixed Point of Asymptotically Nonexpansive Mappings in Banach Spaces*

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Abstract: In this paper, the iteration $x_{n+1} = \alpha_n y + (1 - \alpha_n)T_{i(n)}^{k(n)} x_n$ for a family of asymptotically nonexpansive mappings $T_1, T_2, \cdots, T_N$ is originally introduced in an uniformly convex Banach space. Motivated by recent papers, we prove that under suitable conditions the iteration scheme converges strongly to the nearest common fixed point of the family of asymptotically nonexpansive mappings. The results presented in this paper expand and improve correponding ones from Hilbert spaces to uniformly convex Banach spaces, or from nonexpansive mappings to asymptotically nonexpansive mappings.

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1 Introduction

In this paper, we are interested in the following iteration for a finite family of asymptotically nonexpansive mappings $\{T_1,\ T_2,\ \cdots,\ T_N\}$ in the setting of uniformly convex Banach spaces:

$$x_{n+1} = \alpha_n y + (1 - \alpha_n)T_{i(n)}^{k(n)} x_n, \qquad n \in \mathbf{N}, \tag{1.1}$$

where $T_n = T_{n(\text{mod } N)}$, $n = (k-1)N + i$, and $i = i(n) \in \{1, 2, \cdots, N\}$, $k = k(n) \in \mathbf{N}$, the set of natural numbers.

Especially, if $N = 1$, then (1.1) is reduced to the following iteration

$$x_{n+1} = \alpha_n y + (1 - \alpha_n)T^n x_n, \qquad n \in \mathbf{N}, \tag{1.2}$$

where T is an asymptotically nonexpansive map. Particularly, if T is a nonexpanive mapping and if T^n is seen as T_n, i.e., if $\{T,\ T^2,\ \cdots,\ T^n,\ \cdots\}$ is replaced by an infinite family of

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nonexpansive maps $\{T_1,\ T_2,\ \cdots,\ T_n,\ \cdots\}$, then (1.2) is reduced to the following iteration for an infinite family of nonexpansive maps $\{T,\ T^2,\ \cdots,\ T^n,\ \cdots\}$:

$$x_{n+1} = \alpha_n y + (1-\alpha_n)T_n x_n, \qquad n \in \mathbf{N}, \tag{1.3}$$

which is studied by Chang[1] in the setting of Hilbert spaces.

Theorem 1.1[1] *Let H be a real Hilbert space, C a nonempty closed convex subset of H, and $\{T_n : C \to C,\ n = 1, 2, \cdots\}$ an infinite family of nonexpansive mappings such that $F = \bigcap\limits_{n=1}^{\infty} F(T_n) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a real sequence in $(0, 1)$ such that $\alpha_n \to 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$. For any given $x_1, y \in C$, $\{x_n\} \subset C$ is generated by the iteration (1.3). If there exists a family of nonexpansive mappings $\{G_\gamma\}_{\gamma \in \Gamma}$ such that*

(i) $\bigcap\limits_{\gamma\in\Gamma} F(G_\gamma) \neq \emptyset$ *and* $\bigcap\limits_{\gamma\in\Gamma} F(G_\gamma) \subset \bigcap\limits_{n=1}^{\infty} F(T_n)$;

(ii) $\limsup\limits_{n\to\infty} \|T_n x_n - G_\gamma(T_n x_n)\| = 0$ *for each* $\gamma \in \Gamma$,

then $\{x_n\}$ converges strongly to $Qy \in F = \bigcap\limits_{n=1}^{\infty} F(T_n)$, where Γ is a finite or an infinite index set, and Q is the nearest point projection of H onto F.

Throughout this paper, we assume that E is a uniformly convex Banach space whose norm is uniformly Gâteaux differentiable, C is a nonempty closed convex subset of H, I is the identity mapping, and $F(T) = \{x \in C : x = Tx\}$ is the set of fixed points of mapping T. Denote by $\to$ and $\rightharpoonup$ the strong convergence and weak convergence, respectively.

A mapping $T : C \to C$ is called an asymptotically nonexpansive mapping, if for any $x, y \in C$, there exists a real sequence $\{h_n\}$ such that

$$\|T^n x - T^n y\| \leq h_n \|x - y\|$$

with $h_n \geq 1$ and $\lim\limits_{n\to\infty} h_n = 1$. Especially, if $h_n = 1$ for all $n \in \mathbf{N}$, then $T : C \to C$ is called a nonexpansive mapping.

A mapping $P : E \to C$ is said to be

(1) sunny, if for each $x \in C$ and $t \in [0, 1]$ we have

$$P(tx + (1-t)Px) = Px;$$

(2) a retraction of E onto C, if $Px = x$ for all $x \in C$;

(3) a sunny nonexpansive retraction, if P is a sunny, nonexpansive mapping and a retraction of E onto C.

C is said to be a sunny nonexpansive retract of E, if there exists a sunny nonexpansive retraction of E onto C.

For the sake of the convenience, we may recall the following lemma firstly (see [2] and [3]):

Lemma 1.1 *Let E be a uniformly convex Banach space, C be a nonempty closed convex subset of E and $T : C \to C$ be an asymptotically nonexpansive mapping. Then $I - T$ is semi-closed at zero, i.e., for each sequence $\{x_n\}$ in C, if $\{x_n\}$ converges weakly to $q \in C$ and $\{(I-T)x_n\}$ converges strongly to 0, then $(I-T)q = 0$.*