COMMUNICATIONS
IN
MATHEMATICAL
RESEARCH
27(2)(2011), 156–168

Two Dimensional Cahn-Hilliard Equation with Concentration Dependent Mobility and Gradient Dependent Potential*

HUANG RUI[1], YIN JING-XUE[1] AND WANG LIANG-WEI[2]
(*1. School of Mathematical Sciences, South China Normal University, Guangzhou, 510631*)
(*2. Institute of Mathematics, Jilin University, Changchun, 130012*)

Abstract: In this paper we consider the initial boundary value problem of Cahn-Hilliard equation with concentration dependent mobility and gradient dependent potential. By the L^p type estimates and the theory of Morrey spaces, we prove the Hölder continuity of the solutions. Then we obtain the existence of global classical solutions. The present work can be viewed as an extension to the previous work on the Cahn-Hilliard equation with concentration dependent mobility and potential.

Key words: Cahn-Hilliard equation, concentration dependent mobility, gradient dependent potential
2000 MR subject classification: 35K55, 35G20, 35G30
Document code: A
Article ID: 1674-5647(2011)02-0156-13

1 Introduction

In this paper, we consider the following initial boundary value problem of Cahn-Hilliard equation with concentration dependent mobility and gradient dependent potential:

$$\frac{\partial u}{\partial t} + \operatorname{div}\Big[m(u)\Big(k\nabla\Delta u - \vec{\Phi}(\nabla u)\Big)\Big] = 0, \qquad (x,t)\in Q_T, \tag{1.1}$$

$$\nabla u\cdot\nu\Big|_{\partial\Omega} = \mu\cdot\nu\Big|_{\partial\Omega} = 0, \qquad t\in[0,T], \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad x\in\Omega, \tag{1.3}$$

where Ω is a bounded domain in $\mathbf{R}^2$ with smooth boundary, $Q_T = \Omega\times(0,T)$, ν denotes the unit exterior normal to the boundary $\partial\Omega$, $\mu = k\nabla\Delta u - \vec{\Phi}(\nabla u)$ is the flux, k is a positive constant, $m(u)$ denotes the mobility which depends on the concentration u, and $\vec{\Phi} = (\Phi_1, \Phi_2)$ is a smooth vector function from $\mathbf{R}^2$ to $\mathbf{R}^2$.

The problem (1.1)–(1.3) models many interesting phenomena in mathematical biology, fluid mechanics, phase transition, etc. We refer the readers to [1] for the derivation of the

*Received date: Dec. 20, 2010.
Foundation item: The NSF (11001103) and the SRFDP (200801831002) of China.

equation (1.1) based on the continuum model for epitaxial thin film growth. Recently, such type of equations, especially in the case of one spatial dimension, have interested many mathematicians, for example, [2]–[5]. For the multi-dimensional case of the equation (1.1) with constant mobility, there are also some results obtained recently. For example, Yin and Huang[1] proved the existence and uniqueness of global solutions. Li *et al.*[6] considered the time periodic solutions. But, as far as we know, there are no results about the equation (1.1) with concentration dependent mobility and gradient dependent potential before the present paper. This paper can be viewed as an extension to the previous work [7], where the authors considered the standard Cahn-Hilliard equation with concentration dependent mobility and potential.

The main result of this paper is the following theorem.

Theorem 1.1 *If $u_0(x) \in C^{4+\alpha}(\bar{\Omega})$ for some $0 < \alpha < 1$, $m(s)$ and $\vec{\Phi}(\xi)$ satisfy the following conditions*:

(H1) $m(s) \in C^{1+\alpha}(\mathbf{R})$, $M_1 \le m(s) \le M_2$, *for all* $s \in \mathbf{R}$;

(H2) $\vec{\Phi}(\xi) \in C^{1+\alpha}(\mathbf{R}^2)$, $|\vec{\Phi}(\xi)| \le C|\xi|$, *for all* $\xi \in \mathbf{R}^2$,

where C, M_1 and M_2 are positive constants, then the problem (1.1)–(1.3) *admits a unique classical solution.*

Remark 1.1 Note that in Theorem 1.1, we do not have the smallness restriction on the initial value. This is the main difficulty to prove the Hölder continuity of the solutions, which is a key step to prove the regularity of the solutions. In this paper, we use the a priori L^p estimates combined with the theory of Morrey spaces to overcame this difficulty.

This paper is organized as follows. In the second section, we prove the interior L^p-estimates by energy method. The third section is devoted to proving the L^p-estimates near the boundary. In the last section, we first give the proof of Hölder continuity of the solutions by the L^p-estimates and the theory of Morrey spaces, and then we prove the main result of this paper.

2 The Interior L^p-estimates

In this section, we establish the interior L^p-estimates by the energy method. We first prove the following lemma.

Lemma 2.1 *Suppose the conditions of Theorem* 1.1 *hold. If u is a solution of the problem* (1.1)–(1.3), *then*

$$\sup_{0<t<T} \int_\Omega |\nabla u(x,t)|^2 \mathrm{d}x \le C, \tag{2.1}$$

$$\iint_{Q_T} |\nabla \Delta u|^2 \mathrm{d}x \le C. \tag{2.2}$$