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CONVERGENCE ANALYSIS OF YEE-FDTD SCHEMES FOR 3D MAXWELL'S EQUATIONS IN LINEAR DISPERSIVE MEDIA

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Abstract. In this paper, we develop and analyze finite difference methods for the 3D Maxwell's equations in the time domain in three different types of linear dispersive media described as Debye, Lorentz and cold plasma. These methods are constructed by extending the Yee-Finite Difference Time Domain (FDTD) method to linear dispersive materials. We analyze the stability criterion for the FDTD schemes by using the energy method. Based on energy identities for the continuous models, we derive discrete energy estimates for the FDTD schemes for the three dispersive models. We also prove the convergence of the FDTD schemes with perfect electric conducting boundary conditions, which describes the second order accuracy of the methods in both time and space. The discrete divergence-free conditions of the FDTD schemes are studied. Lastly, numerical examples are given to demonstrate and confirm our results.

Key words. Maxwell's equations, Debye, Lorentz, cold plasma dispersive media, Yee scheme, FDTD method, energy decay, convergence analysis.

1. Introduction

The finite difference time domain (FDTD) method by Kane Yee [50] is a numerical technique for discretizing the time-dependent Maxwell's equations in computational electromagnetics and has been widely used in engineering, physics and computational mathematics [47, 50]. Electromagnetic wave propagation in a material is described by the three dimensional (3D) Maxwell's equations, modeling the evolution in space and time of the electric and magnetic fields, along with constitutive laws, relations between electric and magnetic fluxes and fields, that describe the response of the material to the propagating fields.

The FDTD method was first proposed for a linear dielectric (e.g., free space) by K. S. Yee [50] in 1966, and is also referred to as the (classical) Yee scheme or the Yee-FDTD method. The Yee scheme, as originally constructed, is an explicit scheme for the discretization of the 3D Maxwell's equations on structured staggered space-time grids. The staggered discretization results in a second order accurate method. The classical Yee scheme has been theoretically analyzed for stability and dispersion error [47], convergence analysis and error estimates [40, 41], and has been extended to discretize Maxwell's equations with constitutive laws describing electromagnetic wave propagation in a variety of materials [47].

In this paper, we focus on constitutive laws that do not include any magnetic effects, i.e. the magnetic constitutive law is the same as that in free space (linear dielectric). Previous work in this area includes extension of the Yee scheme to conductive media [47], linear dispersive media [4, 10, 14, 26, 27, 28, 38, 43] using constitutive laws that include models such as the Debye model for orientational polarization [16, 28], Lorentz model for electronic polarization [26, 42], cold plasma model [14, 51, 52] and the Cole-Cole [10, 13] model. In nonlinear optics, nonlinear dispersive models in 1D for the Kerr and Raman effects have been constructed and discretized within this FDTD approach [5, 20, 25]. There is a large literature

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on the construction of Yee type FDTD schemes for other applications including in metamaterials [21, 34], micromagnetics [1, 45], plasmas [15, 39], among others. The classical Yee scheme for a linear dielectric, and many of its extensions, leads to a conditionally stable second order accurate scheme. The scheme may no longer be fully explicit for some constitutive laws, and other complications can arise [5]. In addition, there are extensions of the Yee schemes to higher than second order [5, 6] and to extensions on unstructured meshes [17].

One of the areas in which the Yee scheme has been relatively less studied is convergence analysis, while there are several papers on dispersion analysis of the Yee and Yee type schemes. In our recent work [7], we presented Yee schemes and their convergence analysis for the 2D Maxwell's equations in Debye and Lorentz (linear) dispersive media. The Yee scheme for both media was proved to be conditionally stable under the same stability condition as the classical Yee scheme. We proved that the proposed Yee scheme in both media is of second order convergent in time and space by the energy method.

In this paper, we extend the convergence of Yee schemes for linear dispersive media in two spatial dimensions to 3D. We consider three types of models for linear dispersive materials; the (single pole) Debye model for orientational polarization (Maxwell-Debye), the (single pole) Lorentz (Maxwell-Lorentz) and the isotropic cold plasma (Maxwell-Cold Plasma) model. We focus on the construction and analysis of the finite difference time domain methods based on the staggered Yee grids for 3D Maxwell's equations in these three linearly dispersive media. We show that our fully discrete schemes are conditionally stable via the energy method, and convergent with second order accuracy. Moreover, we use the energy technique to analyze the discrete divergence for the discrete Maxwell's equations in dispersive media. The energy method is a powerful method used on both the continuous PDEs and discrete finite difference methods by defining an energy associated with the solution and then showing that the energy is non-increasing. Recently, the energy technique has been applied for analyzing stability and convergence properties of the Yee scheme in various dispersive media, applied in operator splitting FDTD methods [8, 9, 11, 18, 49, 35, 36], finite element methods (FEM) and discontinuous Galerkin (DG) methods (for example see [22, 29, 30, 31, 32, 33, 37, 48] and references therein).

We present numerical experiments to illustrate our theoretical results by constructing exact solutions for Maxwell's equations in these linearly dispersive media: Debye, Lorentz and Cold plasma. We also investigate the discrete divergence properties of electric and magnetic flux densities for these dispersive media. Our analysis shows that the numerical divergence satisfies discrete versions of the continuous Gauss's laws for the 3D Maxwell's equations in dispersive media.

This paper is organized as follows. In Section 2, we present the 3D Maxwell's equations in three types of linearly dispersive media (Debye, Lorentz and isotropic cold plasma) and then present their corresponding weak formulations. In addition, we present energy decay results for these dispersive models that are available in the literature [7, 31]. Section 3 details the staggered discretization in space and time. The stability, discrete energy estimates and convergence analysis, including the analysis of the discrete divergence property are presented in Sections 4, 5 and 6. Numerical experiments demonstrating our theoretical results are presented in Section 7. We provide concluding remarks in Section 8.