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FAST GAUSS-RELATED QUADRATURE FOR HIGHLY OSCILLATORY INTEGRALS WITH LOGARITHM AND CAUCHY-LOGARITHMIC TYPE SINGULARITIES

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Abstract. This paper presents an efficient method for the computation of two highly oscillatory integrals having logarithmic and Cauchy-logarithmic singularities. This approach first requires the transformation of the original oscillatory integrals into a sum of line integrals with semi-infinite intervals. Afterwards, the coefficients of the three-term recurrence relation that satisfy the orthogonal polynomial are obtained by using the method based on moments, where classical Laguerre and Gautschi's logarithmic weight functions are employed. The algorithm reveals that with fixed n, the method is capable of achieving significant figures within a short time. Furthermore, the approach yields higher accuracy as the frequency increases. The results of numerical experiments are given to substantiate our theoretical analysis.

Key words. Highly oscillatory integrals, modified Chebyshev algorithm, steepest descent method, Cauchy principal value integrals, logarithmic weight function, algebraic and logarithm singular integrals.

1. Introduction

Logarithmic singular integrals have numerous applications in wave scattering, diffraction problems, aero and hydroacoustic problems, elasticity problems [1, 2, 3], etc. Boundary Element Methods, abbreviated as BEMs, are some of the popular methods for handling partial differential equations involving the singular boundary integral equations. Nevertheless, when the integral contains simultaneous integrands with singular factors and a very large frequency of oscillation, traditional numerical methods fail to achieve numerical accuracy for these types of integrals. Consequently, three kinds of singularities arise: (i) Weak singularity (ii) Strong singularity and (iii) Hyper singularity.

In this paper, we focus on the efficient computation of two highly oscillatory integrals having logarithmic and Cauchy-logarithmic singularities of the forms

$$(1)\qquad Q\left[f\right] := \int_a^b e^{ikx}\left(x-a\right)^{\alpha}\left(b-x\right)^{\beta}\left|x-c\right|^{\gamma}\ln\left(x-a\right)\ln\left(b-x\right)f\left(x\right)dx,$$

and

$$(2)\qquad Q'\left[f\right] := \int_a^b \frac{e^{ikx}\left(x-a\right)^{\alpha}\left(b-x\right)^{\beta}\ln\left(x-a\right)\ln\left(b-x\right)f\left(x\right)}{\left(x-\rho\right)}dx,$$

where $\alpha, \beta, \gamma > -1$, $-\infty < a < c < b < +\infty$, $|k| >> 1$, $i^2 = -1$, $a < \rho < b$ and $f\left(x\right)$ is a holomorphic function in an appropriate complex region consisting $[a, b]$. While observing the integral (1), one encounters many types of logarithmic oscillatory integrals such as

(1) $\int_a^b e^{ikx}\left(x-a\right)^{\alpha}\left(b-x\right)^{\beta}\ln\left(x-a\right)f\left(x\right)dx$,
(2) $\int_a^b e^{ikx}\left(x-a\right)^{\alpha}\left(b-x\right)^{\beta}\ln\left(x-a\right)\ln\left(b-x\right)f\left(x\right)dx$,

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which, previously, have been discussed extensively in [4] and [5]. The authors applied the truncated Chebyshev expansion for approximating the given smooth function, then computed the singular part using non-homogeneous recurrence relations of modified moments. However, their approach requires a lot of computation time and function evaluation when a very large value of frequency is applied, due to the error analysis on k (frequency) and stability of the recurrence relations. In the example section, we show the accuracy of the proposed method for efficient computation of the aforementioned types of integrals.

Normally, highly oscillatory integrands have the form of $H(x)\, e^{ikg(x)}$, where k is strictly large and well known as the wave number, or simply, as the frequency of oscillation, and where $H(x)$ and $g(x)$ are amplitude and phase functions, respectively. Moreover, $H(x)$ may contain singularities of weak and strong types whereas $g(x)$ has stationary points of a certain order. Generally, evaluation of highly oscillatory integrals is considered as a challenging task, in particular, the logarithmic types. Due to their wide range of applications in various branches of mathematics, applied computational sciences, and other areas of applied science and technology such as electromagnetic scattering, image processing, quantum mechanics, astronomy, seismology, many physical problems; phenomenal methods have been developed for solving oscillatory integrals. Much emphasis has been put in obtaining the best approximation of algebraic singular integrals. For instance, the integrals of types $\int_a^b e^{i\omega x} f(x)\, dx$ and $\int_a^b e^{i\omega x} (x-a)^{\alpha} (b-x)^{\beta} f(x)\, dx$, for more details, one can refer to [6, 7, 8, 9, 10, 11, 12, 13, 14, 30, 31], and the references therein. The integral type (2) exhibits severe shortcomings inasmuch as its integrands involve not only oscillatory but also weak and strong (Cauchy type) singularities. These integrals are also well known as Cauchy-type integrals and have recently become of great interest in the computational community. Commonly, the Cauchy principal value integral of type $\int_a^b f(x)/(x-\rho)\, dx$ with $-\infty \le a < b \le \infty$ such that $a < \rho < b$, is being recognized as Hilbert transformation. The sufficient condition for the existence of the Hilbert integral transform is that $f(x)$ satisfies Lipschitz and Hölder conditions. The integral (2) presents different types of Cauchy oscillatory integrals of which the following can be mentioned:

(1) $\int_a^b e^{ikx} \frac{f(x)}{x-\rho} dx$,
(2) $\int_a^b \frac{e^{ikx}(x-a)^{\alpha}(b-x)^{\beta}\ln(x-a) f(x)}{x-\rho} dx$.

When integrals of the above types arise, the classical Gauss rules cannot directly be applied, due to the fact that the integrands become unbounded at $x = \rho$. Special treatment is needed for their evaluation with a high order numerical accuracy. As a plausible solution, we propose a fast and accurate numerical method.

Herein, integrands are considered to be analytic in an appropriately large region containing the interval of integration. We then employ the Cauchy integral approach to transform integrals (1) and (2) into a sum of several lines integrals with the semi-infinite interval $[0, +\infty)$. Subsequently, we construct a Gaussian-type quadrature rule. While constructing this rule, we utilize a special type of weight function known as Gautchi-Logarithmic weight function [15]. Nonetheless, we use the method-based moments, the modified Chebyshev algorithm, and Jacobi matrix for the efficient computation of nodes and weights used for the construction of the related quadrature rule.

The rest of the paper is structured as follows: In the next section we evaluate the integral (1) and (2) using the Cauchy integral theorem approach. In Section 3, we construct the Gauss-type quadrature rule. Section 4 is dedicated to the numerical