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Dissipative and Conservative Local Discontinuous Galerkin Methods for the Fornberg-Whitham Type Equations

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Abstract. In this paper, we construct high order energy dissipative and conservative local discontinuous Galerkin methods for the Fornberg-Whitham type equations. We give the proofs for the dissipation and conservation for related conservative quantities. The corresponding error estimates are proved for the proposed schemes. The capability of our schemes for different types of solutions is shown via several numerical experiments. The dissipative schemes have good behavior for shock solutions, while for a long time approximation, the conservative schemes can reduce the shape error and the decay of amplitude significantly.

AMS subject classifications: 65M60, 35L75, 35G25

Key words: Discontinuous Galerkin method, Fornberg-Whitham type equation, dissipative scheme, conservative scheme, error estimates.

1 Introduction

The Fornberg-Whitham type equation we study in this paper is given by

$$u_t+f(u)_x+(1-\partial_x^2)^{-1}u_x=0, \quad I\in[a,b], \quad t>0 \tag{1.1}$$

or its equivalent form

$$u_t-u_{xxt}+f(u)_x+u_x=f(u)_{xxx}, \tag{1.2}$$

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by operating $(1-\partial_x^2)$ on (1.1). We consider the nonlinear term $f(u)=\frac{1}{p}u^p$, where $p\geq 2$ is an integer. When the parameter $p=2$, Eq. (1.1) becomes the Fornberg-Whitham equation derived in [21] as a nonlinear dispersive wave equation. There are three conservative quantities for the Fornberg-Whitham type equation

$$E_0=\int_I u dx, \quad E_1=\int_I (u-u_{xx})dx, \quad E_2=\int_I u^2 dx, \tag{1.3}$$

where the quantity E_0 is called mass, and E_2 is energy.

Many mathematical properties of the Fornberg-Whitham equation have been discussed, this equation was first proposed for studying the qualitative behavior of wave breaking in [21]. Some investigation of wave breaking conditions can be found in [12,14]. Note that the Fornberg-Whitham equation is also called Burgers-Poisson equation in [8]. There has been lots of work focusing on finding the traveling wave solutions in [41,42]. It admits a wave of greatest height, as a peaked limiting form of the traveling wave solution [7]. Recently, some well-posedness results are proposed in [10,11]. There are not many numerical schemes for the Fornberg-Whitham type equation. In [13], the finite difference method is adopted to solve the shock solution. The authors did some valuable numerical analysis by the discontinuous Galerkin method in [16], in which the comparisons has been made between the conservative scheme and dissipative scheme, as well as theoretical analysis.

The discontinuous Galerkin (DG) method was first introduced by Reed and Hill in 1973 [18] for solving steady-state linear hyperbolic equations. The key point of this method is the design of suitable inter-element boundary treatments (so-called numerical fluxes) to obtain highly accurate and stable schemes in several situations. Within the DG framework, the local discontinuous Galerkin (LDG) method can be obtained by extending to handle derivatives of order higher than one. The first LDG method was introduced by Cockburn and Shu in [5] for solving the convection-diffusion equation. Their work was motivated by the successful numerical experiments of Bassi and Rebay [1] for compressible Navier-Stokes equations. The LDG methods can be applied in many equations, such as KdV type equations [15,26–28,32,39], Camassa-Holm equations [24,35], Degasperis-Procesi equation [31], Schrödinger equations [23,25], and more nonlinear equations or system [17,24,30,33,34].

There are also many conservative DG schemes that are proposed to "preserve structure", such as KdV equation [2, 15, 39], Zakharov system [22], Schrödinger-KdV system [23], short pulse equation [40], etc. Usually, the structure preserving schemes can help reduce the shape error of waves along with long time evolution. For example in [2, 15, 16, 39], compared with dissipative schemes, the energy conservative or Hamiltonian conservative numerical schemes for the KdV equation have less shape error or amplitude damping for long time approximations, especially in the low-resolution cases.

In this paper, we adopt the LDG method as a spatial discretization to construct high order accurate numerical schemes for the Fornberg-Whitham type equations. Through the two equivalent forms of this type of equation, we develop dissipative and conserva-