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Little Hankel Operators on the Weighted Bergman Space of the Unit Ball*

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Abstract: In this paper we mainly consider little Hankel operators with square-integrable symbols on the weighted Bergman spaces of the unit ball. We obtain that Schatten class of little Hankel operators is equivalent to Schatten class of positive Toeplitz operators under the conditions that $SMO(f) \in L^{\frac{p}{2}}(B_n, \mathrm{d}\lambda)$ and $2 \leq p < \infty$, which is very important to research the relation between Toeplitz operators and little Hankel operators.

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1 Introduction

Let B_n denote the unit ball of $\mathbb{C}^n$. Let v be the Lebesgue measure on B_n. For $-1 < \alpha < \infty$, we denote by v_α the normalized measure on B_n defined by

$$\mathrm{d}v_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha \mathrm{d}v(z),$$

where c_α is the normalizing constant such that

$$v_\alpha(B_n) = 1.$$

For $z \in B_n$, let φ_z be the automorphism of B_n with

$$\varphi_z(0) = z, \qquad (\varphi_z)^{-1} = \varphi_z.$$

The mapping φ_z is described in [1].

The weighted Bergman space $A^2_\alpha(B_n)$ is the space of holomorphic functions which are square-integrable with respect to $\mathrm{d}v_\alpha$ on B_n. The reproducing kernel in $A^2_\alpha(B_n)$ is given by

$$K_z^{(\alpha)}(\omega) = \frac{1}{(1 - \langle \omega,\ z\rangle)^{n+\alpha+1}}, \qquad z, \omega \in B_n.$$

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If $\langle\cdot,\ \cdot\rangle_\alpha$ denotes the inner product on $L^2(B_n, \mathrm{d}v_\alpha)$, then

$$\langle h,\ K_z^{(\alpha)}\rangle_\alpha = h(z), \qquad h \in A_\alpha^2(B_n),\ z \in B_n.$$

In this paper we use $\|\cdot\|_p$ to denote the norm on $L^p(B_n,\ \mathrm{d}v_\alpha)$.

The orthogonal projection P_α from $L^2(B_n,\ \mathrm{d}v_\alpha)$ onto $A_\alpha^2(B_n)$ is given by

$$(P_\alpha g)(z) = \langle g,\ K_z^{(\alpha)}\rangle_\alpha = \int_{B_n} \frac{g(\omega)}{(1-\langle z,\omega\rangle)^{n+\alpha+1}}\mathrm{d}v_\alpha(\omega).$$

Since H^∞, the space of bounded analytic functions on B_n, is dense in $A_\alpha^2(B_n)$, for $f \in L^2(B_n,\ \mathrm{d}v_\alpha)$, the Toeplitz operator T_f and Hankel operator H_f on $A_\alpha^2(B_n)$ are densely defined by

$$T_f g(z) = \int_{B_n} \frac{f(\omega)g(\omega)}{(1-\langle z,\ \omega\rangle)^{n+\alpha+1}}\mathrm{d}v_\alpha(\omega)$$

and

$$H_f g(z) = \int_{B_n} \frac{f(z)g(z)-f(\omega)g(\omega)}{(1-\langle z,\ \omega\rangle)^{n+\alpha+1}}\mathrm{d}v_\alpha(\omega).$$

Let $\overline{A_\alpha^2(B_n)}$ be the space of conjugate holomorphic functions in $L^2(B_n,\ \mathrm{d}v_\alpha)$. It is easy to prove that $\overline{A_\alpha^2(B_n)} = \{\bar f : f \in A_\alpha^2(B_n)\}$ is closed in $L^2(B_n,\ \mathrm{d}v_\alpha)$. The orthogonal projection $\overline{P_\alpha}$ from $L^2(B_n,\ \mathrm{d}v_\alpha)$ onto $\overline{A_\alpha^2(B_n)}$ is given by

$$\overline{P_\alpha} f(z) = \int_{B_n} \frac{f(\omega)}{(1-\langle\omega, z\rangle)^{n+\alpha+1}}\mathrm{d}v_\alpha(\omega), \qquad f \in L^2(B_n,\ \mathrm{d}v_\alpha).$$

For $f \in L^2(B_n,\ \mathrm{d}v_\alpha)$ and $z \in B_n$, the little Hankel operator

$$h_f : A_\alpha^2(B_n) \to \overline{A_\alpha^2(B_n)}$$

is densely defined by

$$h_f g(z) = \int_{B_n} \frac{f(\omega)g(\omega)}{(1-\langle\omega,\ z\rangle)^{n+\alpha+1}}\mathrm{d}v_\alpha(\omega).$$

In the study of little Hankel operator, it turns out that it is more convenient to study $h_{\bar f}$ instead of h_f. This is perhaps justified by the use of the projection $\overline{P_\alpha}$ in the definition of little Hankel operators.

Let S_p denote the set of Schatten class of operators on $A_\alpha^2(B_n)$, that is, the set of all bounded linear operators $F : A_\alpha^2(B_n) \to H$, where H is a Hilbert space, such that $(F^*F)^{\frac{p}{2}}$ is in the set of trace class of operators on $A_\alpha^2(B_n)$.

In the past few decades many mathematicians have paid more attention to the study of concrete operators on the function spaces, such as Toeplitz operator, Hankel operator and little Hankel operator. For the boundedness of such operators there have obtained a lot of results, which can be found in [2]–[6].

Schatten class of operators is one of the most widely studied classes of concrete operators. The study of Schatten class of operators on the Bergman space has generated an extensive list of results in the operator theory and in the theory of function spaces. Zhu[7] has proven that Hankel operators H_f and $H_{\bar f}$ on the Bergman space of the unit ball are in S_p if and only if the mean oscillation of f is in $L^p(B_n,\ \mathrm{d}\lambda)$, where $2 \le p < \infty$ and f is in $L^2(B_n,\ \mathrm{d}v)$. Lü and Xu[8] have extended Zhu's results to the weighted Bergman space of the unit ball. Liu[9] has given the necessary and sufficient conditions for the boundedness and compactness of little