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# Conservative Numerical Schemes for the Nonlinear Fractional Schrödinger Equation

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**Abstract.** This paper deals with the Crank-Nicolson Fourier collocation method for the nonlinear fractional Schrödinger equation containing a fractional derivative. We prove that at each discrete time the method preserves the discrete mass and energy conservation laws. The existence, uniqueness and convergence of the numerical solution are also investigated. In particular, we show that the method has the second-order accuracy in time and the spectral accuracy in space. Since the proposed schemes are implicit, they are solved by an iteration algorithm with FFT. Two examples illustrate the efficiency and accuracy of the numerical schemes.

**AMS subject classifications**: 65M12, 65M70

**Key words**: Crank-Nicolson Fourier collocation method, nonlinear fractional Schrödinger equation, conservation laws, existence and uniqueness, convergence.

## 1. Introduction

The well-known Schrödinger equations have a wide range of applications in quantum mechanics, plasma, nonlinear optics, hydrodynamics and other fields [6,16,17,23]. On the other hand, in recent years fractional differential equations attracted considerable attention in sciences and engineering [8,11,28]. In particular, since the fractional Schrödinger equations describe the motion law of particles, the corresponding energy can be used in the study of microscopic systems. These fractional equations describe the change of the quantum state of a physical systems more accurately than the equations of integer order. Therefore, they became one of important tools in quantum mechanics. In this work, we consider the following nonlinear fractional Schrödinger equation

$$\begin{aligned} &i\partial_t\varphi+\frac{1}{\alpha}|\partial_z|^\alpha\varphi-\lambda|\varphi|^2\varphi=0, \quad t\in(0,T],\\ &\varphi(z,0)=\varphi_0(z), \qquad\qquad\qquad z\in\Omega,\\ &\varphi(a,t)=\varphi(b,t), \qquad\qquad\qquad t\in(0,T], \end{aligned} \tag{1.1}$$

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where $0<\alpha<1$, $\Omega=[a,b]$, $i$ is the imaginary unit, $\lambda$ a given real constant, $\varphi$ a scalar complex-valued nucleon field, and $\varphi_0$ a given complex-valued function. The fractional derivative is defined by $|\partial_z|^\alpha=\mathscr{F}^{-1}|k|^\alpha\mathscr{F}$, with $\mathscr{F}$ and $\mathscr{F}^{-1}$ denoting the direct and inverse Fourier transforms, respectively. If $\alpha=1/2$, the above equation is used to solve the water wave equation [10]. As a matter of fact, it is easy to obtain that the solution of (1.1) satisfies the conservative laws

$$Q(t)=\int_\Omega|\varphi(z,t)|^2dz=Q(0),$$
$$E(t)=\int_\Omega\left(\frac{1}{\alpha}\big(|\partial_z|^\alpha\varphi(z,t)\big)\varphi(z,t)-\frac{\lambda}{2}|\varphi(z,t)|^4\right)dz=E(0).$$

These laws yield the boundedness of $\|\varphi\|$. Naumkin [18] provided conditions for the unique solvability of (1.1). In particular, if $\varphi_0\in H^3\cap H^{2,1}$ has a sufficiently small norm $\|\varphi_0\|_{H^3\cap H^{2,1}}$, then $\varphi\in C((0,+\infty);H^3\cap H^{2,1})$ and $\|\varphi\|_\infty\le C\|\varphi_0\|_{H^3\cap H^{2,1}}$, where $H^{s,m}$ is the weighted Sobolev space

$$H_p^{s,m}:=\left\{\varphi\in L^p:\|\varphi\|_{H_p^{s,m}}=\|\langle z\rangle^m\langle i\partial_z\rangle^s\varphi\|_{L^p}<\infty\right\},\quad m,s\in\mathbb{R},$$

where $\langle z\rangle=\sqrt{1+z^2}$, $\langle i\partial_z\rangle=\sqrt{1-\partial_z^2}$. If $p=2$, we write $H^{s,m}$ for $H_2^{s,m}$.

It is well-known that analytic solutions of Schrödinger equations, especially fractional ones, can be rarely found. Therefore, there are multifarious numerical methods developed for their solution, including finite difference methods [4, 25, 30], finite element methods [3,12,19], discontinuous Galerkin [2,20,24], spectral and pseudo-spectral methods [5,14, 15,26,29], etc.

Spectral methods are high-order methods widely used in differential equations. However, in order to obtain good approximations for fractional differential equations, the methods have to be carefully chosen. Wang and Tang [13] proposed a time-space spectral method with spectral accuracy both in time and space. For bounded and unbounded domains, Shen and Wang [22] used classical orthogonal polynomials with hyperbolic cross approximation. Zeng *et al.* [28] employed the Crank-Nicolson Galerkin-Legendre spectral method to the two-dimensional fractional nonlinear reaction-diffusion equation and discussed the discretisation of the Riesz fractional derivative. Huang *et al.* [9] considered the combination of a spectral Galerkin method in space and a second order finite difference method in time for space fractional diffusion equations. They also proved the stability of the method and derived error estimates for the corresponding temporal semi-discrete scheme.

We note that spectral collocation methods are easier to implement than Galerkin spectral method, especially in the case of nonlinear systems. For periodic problems, it is natural to use Fourier series. Many practical physical problems such as crystal structures and homogeneous turbulence are periodic. In this paper, we consider the Crank-Nicolson Fourier collocation (CNFC) method for nonlinear fractional Schrödinger equation with periodic boundary conditions. It is easily implementable and provides fast calculations.