Communications
in
Mathematical
Research
26(3)(2010), 269–279

Affine Locally Symmetric Surfaces in $\mathbf{R}^{4*}$

Hou Zhong-hua and Fu Yu
(*School of Mathematical Sciences, Dalian University of Technology, Dalian, Liaoning, 116024*)

Abstract: The nondegenerate affine locally symmetric surfaces in $\mathbf{R}^4$ with the transversal bundle defined by Nomizu and Vrancken[1] have been studied and a complete classification of the locally symmetric surfaces with flat normal bundle has been given.
Key words: locally symmetric surface, flat normal bundle, equiaffine normal bundle
2000 MR subject classification: 53A15
Document code: A
Article ID: 1674-5647(2010)03-0269-11

1 Introduction

Let M^2 be a nondegenerate affine surface immersed in the affine space $\mathbf{R}^4$ with the standard affine connection D. An equiaffine structure was found by Nomizu and Vrancken[1] and further investigations showed that the construction leads to natural geometric properties. For the construction of this equiaffine normal plane bundle and the notations used throughout this paper, we refer to [1] and [2].

It is well known that a torsion-free connection ∇ is locally symmetric if and only if its curvature tensor satisfies the condition $\nabla R = 0$. Locally symmetric surfaces in three dimensional affine space $\mathbf{R}^3$ have been studied by a number of authors (see [3]–[6] etc.). In the present paper, we study locally symmetric surfaces in four-dimensional affine space $\mathbf{R}^4$. Magid and Vrancken[2] showed a complete classification of all surfaces with flat induced connection and flat normal bundle. Based on their work, we classify all the nonflat locally symmetric affine surfaces with flat normal connection in this paper. Furthermore, we give three examples of locally symmetric surfaces with flat normal connection, and straightforward computations moreover show that they are not flat. Precisely, we prove

Theorem 1.1 *Let M^2 be a nondegenerate affine locally symmetric surface in $\mathbf{R}^4$ with flat normal connection. Then M^2 is either flat or affinely equivalent to an open part of one of the following surfaces*:

*Received date: Feb. 23, 2009.

(1) *One of the surfaces*

$$f(u,v) = (\mathrm{e}^{au}\cos v,\ \mathrm{e}^{au}\sin v,\ y_1(u),\ y_2(u)),$$
$$f(u,v) = (\mathrm{e}^{au}\cosh v,\ \mathrm{e}^{au}\sinh v,\ y_1(u),\ y_2(u)),$$

where

$$u \mapsto \gamma(u) = (y_1(u), y_2(u))$$

is an arbitrary planar curve parameterized so that

$$|\gamma'\gamma''| = \mathrm{e}^{au},$$

and a is a nonzero constant;

(2) *The surface*

$$f(u,v) = \alpha(v)\mathrm{e}^{au} + \beta u,$$

where α is a curve in $\mathbf{R}^4$ satisfying

$$\alpha'''(v) = \varphi(v)\alpha + \psi(v)\alpha',$$

in which φ and ψ are functions depending only on v, β is a constant vector in $\mathbf{R}^4$ selected so that

$$|\alpha\alpha'\alpha''\beta| = 1,$$

and a is a nonzero constant;

(3) *The surface*

$$f(u,v) = (\mathrm{e}^{au}v^2 + \varphi(u),\ \mathrm{e}^{au}v,\ \mathrm{e}^{au}, u),$$

where φ is a function depending on u and a is a nonzero constant.

2 Preliminaries

Here, we recall the basic equations for nondegenerate affine surfaces in $\mathbf{R}^4$. For more details and proofs, see [1]. Let M^2 be a surface in $\mathbf{R}^4$ and let $u = \{X_1, X_2\}$ be a local differentiable frame on a neighborhood U of a point p of M^2. We introduce a symmetric bilinear form G_u by

$$2G_u(Y,Z) = [X_1, X_2, D_Y X_1, D_Z X_2] + [X_1, X_2, D_Z X_1, D_Y X_2], \tag{2.1}$$

where D is the standard flat connection and $[*,*,*,*]$ is the usual determinant on $\mathbf{R}^4$. It is easy to see that G_u is nondegenerate is independent of the choice of the local frame u. Hence, we call M^2 nondegenerate if G_u is nondegenerate. From now on, we only consider nondegenerate surfaces in $\mathbf{R}^4$, following the approaches in [1]. If M^2 is nondegenerate, then the affine metric g is defined by

$$g(Y,Z) = \frac{G_u(Y,Z)}{(\det_u G_u)^{\frac{1}{3}}}\ , \tag{2.2}$$

where the determinant is calculated with respect to the frame u. If g is negative definite, by interchanging two coordinates in $\mathbf{R}^4$, then we can always make g positive definite. So, if g is definite, then we always assume that g is positive definite.

A plane bundle σ is called transversal if, together with the tangent plane, it spans the