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Weighted Estimates for the Maximal Commutator of Singular Integral Operator on Spaces of Homogeneous Type*

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Abstract: Weighted estimates with general weights are established for the maximal operator associated with the commutator generated by singular integral operator and BMO function on spaces of homogeneous type, where the associated kernel satisfies the Hölder condition on the first variable and some condition which is fairly weaker than the Hölder condition on the second variable.

Key words: spaces of homogeneous type, weighted estimates, singular integral operator, commutator, maximal operator

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1 Introduction

We work on a space of homogeneous type. Let $\mathscr{X}$ be a set endowed with a positive Borel regular measure μ and a symmetric quasi-metric d satisfying that there exists a constant $\kappa \ge 1$ such that for all $x, y, z \in \mathscr{X}$,

$$d(x, y) \le \kappa[d(x, z) + d(z, y)].$$

The triple $(\mathscr{X}, d, \mu)$ is said to be a space of homogeneous type in the sense of Coifman and Weiss[1], if μ satisfies the following doubling condition: there exists a constant $C \ge 1$ such that for all $x \in \mathscr{X}$ and $r > 0$,

$$\mu(B(x, 2r)) \le C\mu(B(x, r)).$$

Moreover, if C is the smallest constant for which the measure μ verifies the doubling condition, then $D = \log_2 C$ is called the doubling order of μ and we have

$$\frac{\mu(B_1)}{\mu(B_2)} \le C_\mu \Big(\frac{r_{B_1}}{r_{B_2}}\Big)^D \qquad \text{for all balls } B_2 \subset B_1 \subset \mathscr{X},$$

where r_{B_i} denotes the radius of B_i, $i = 1, 2$.

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We remark that although all balls defined by d satisfy the axioms of complete system of neighborhood in $\mathscr{X}$, and therefore induce a topology in $\mathscr{X}$, the balls $B(x, r)$ for $x \in \mathscr{X}$ and $r > 0$ need not to be open with respect to this topology. However, by a remarkable result of Macías and Segovia in [2], we know that there exists another quasi-metric $\tilde{d}$ which is equivalent to d such that the balls corresponding to $\tilde{d}$ are open in the topology induced by $\tilde{d}$. Thus, throughout this paper, we assume that the balls $B(x, r)$ for $x \in \mathscr{X}$ and $r > 0$ are open.

Let T be a linear $L^2(\mathscr{X})$-bounded operator with kernel K in the sense that for all $f \in L^2(\mathscr{X})$ with bounded support and almost all $x \notin \operatorname{supp} f$,

$$Tf(x) = \int_{\mathscr{X}} K(x,y)f(y)\mathrm{d}\mu(y), \tag{1.1}$$

where K is a locally integrable function on $\mathscr{X} \times \mathscr{X} \backslash \{(x,y) : x = y\}$. For $b \in \mathrm{BMO}(\mathscr{X})$, define the commutator generated by T and b by

$$T_b f(x) = b(x)Tf(x) - T(bf)(x), \qquad f \in L_0^\infty(\mathscr{X}), \tag{1.2}$$

where and in the following, $L_0^\infty(\mathscr{X})$ denotes the set of bounded functions with bounded support. The maximal operator associated with the commutator T_b is defined by

$$T_b^* f(x) = \sup_{\epsilon>0} |T_{\epsilon;\, b} f(x)|, \tag{1.3}$$

where

$$T_{\epsilon;\ b} = b(x)T_\epsilon f(x) - T_\epsilon(bf)(x), \qquad f \in L_0^\infty(\mathscr{X}),$$

and T_ϵ ($\epsilon > 0$) is the truncated operator defined by

$$T_\epsilon f(x) = \int_{d(x,y)>\epsilon} K(x,\, y)f(y)\mathrm{d}\mu(y).$$

The operator T_b^* has been considered by many authors. When the associated kernel K satisfies the size condition

$$|K(x,y)| \le \frac{C}{\mu(x, d(x,y))}, \qquad x, y \in \mathscr{X},\ x \ne y \tag{1.4}$$

and the Hölder smoothness conditions

$$|K(x,y) - K(x,y')| \le C\frac{(d(y,y'))^\eta}{\mu(B(y, d(x,y)))(d(x,y))^\eta}, \qquad \text{if } d(x,y) \ge 2d(y,y') \tag{1.5}$$

and

$$|K(y,x) - K(y',x)| \le C\frac{(d(y,y'))^\eta}{\mu(B(y, d(x,y)))(d(x,y))^\eta}, \qquad \text{if } d(x,y) \ge 2d(y,y') \tag{1.6}$$

with some $\eta \in (0,1]$, for the operator T_b^*, Hu and Wang[3] proved L^p weighted estimates with general weight, Hu *et al.*[4] established weighted endpoint estimates with general weight. Whether one of the smoothness conditions can be replaced by the weaker one, it is of considerable interest. To state our result, we first give some notations.

Let E be a measurable set with $\mu(E) < \infty$. For any fixed $l > 0$ and a suitable function f, set

$$\|f\|_{L(\lg L)^l,\, E} = \inf\Big\{\lambda > 0 : \frac{1}{\mu(E)}\int_E \frac{|f(x)|}{\lambda} \lg^l \Big(\mathrm{e} + \frac{|f(x)|}{\lambda}\Big)\mathrm{d}\mu(x) \le 1\Big\}.$$

The maximal operator $\mathcal{M}_{L(\lg L)^l}$ is defined by

$$\mathcal{M}_{L(\lg L)^l} f(x) = \sup_{B \ni x} \|f\|_{L(\lg L)^l,\, B},$$