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# Generating Layer-Adapted Meshes Using Mesh Partial Differential Equations

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**Abstract.** We present a new algorithm for generating layer-adapted meshes for the finite element solution of singularly perturbed problems based on mesh partial differential equations (MPDEs). The ultimate goal is to design meshes that are similar to the well-known Bakhvalov meshes, but can be used in more general settings: specifically two-dimensional problems for which the optimal mesh is not tensor-product in nature. Our focus is on the efficient implementation of these algorithms, and numerical verification of their properties in a variety of settings. The MPDE is a nonlinear problem, and the efficiency with which it can be solved depends adversely on the magnitude of the perturbation parameter and the number of mesh intervals. We resolve this by proposing a scheme based on $h$-refinement. We present fully working FEniCS codes [Alnaes *et al.*, Arch. Numer. Softw., 3 (100) (2015)] that implement these methods, facilitating their extension to other problems and settings.

**AMS subject classifications**: 65N30, 65N50, 65-04

**Key words**: Mesh PDEs, finite element method, PDEs, singularly perturbed, layer-adapted meshes.

## 1. Introduction

### 1.1. Motivation and outline

Differential equations arise in many areas of science and engineering. They model how phenomena change over time or space, such as how a cancerous tumour grows, how a string vibrates, or how a pollutant disperses. So their solution is of fundamental importance. Analytical solutions are rarely available, and so one must rely on approximate solutions, obtained using numerical schemes."Singularly perturbed" differential

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equations (SPDEs) have a small positive constant, usually denoted $\varepsilon$, multiplying the leading term. Solutions to SPDEs typically exhibit boundary layers (and may also contain interior layers). Resolving these layers is a primary concern of the numerical solution of SPDEs. This is usually achieved using specialised layer-adapted meshes. Our goal is to present a novel method to efficiently generate such meshes, and so compute qualitatively and quantitatively accurate numerical solutions to SPDEs.

The SPDEs that we focus on in this paper are elliptic reaction-diffusion problems of the form

$$-\varepsilon^2 \Delta u + ru = f \quad \text{in} \quad \Omega \subseteq \mathbb{R}^d \quad \text{with} \quad u|_{\partial\Omega} = 0, \tag{1.1}$$

where $d = 1, 2$. Here, $r$ and $f$ are given (smooth) functions, $\varepsilon > 0$ and $r \geq \beta^2$ on $\overline{\Omega}$, where $\beta$ is some positive constant. The so-called "reduced problem" for (1.1) is obtained by formally setting $\varepsilon = 0$ and neglecting the boundary conditions; in the case of (1.1), this is $u_0 = f/r$. The solution to (1.1) will feature layers near boundaries where $u_0$ does not agree with the boundary conditions. Note that (1.1) is singularly perturbed in the sense that it is well-posed for all non-zero values of $\varepsilon$, but ill-posed if one formally sets $\varepsilon = 0$. (For a more formal definition see, e.g., [21, p. 2] or [14, p. 2-3]).

To ensure the layer regions of solutions to (1.1) are resolved, the local mesh width needs to be very small, in a way that depends on $\varepsilon$. Since we are interested in the case where $\varepsilon \ll 1$, uniform meshes are completely unsuitable; highly non-uniform layer-adapted meshes are preferred [21].

The novel aspect of this article is our proposal to use "Mesh PDEs" (MPDEs) to generate suitable layer-adapted meshes. We present algorithms to generate these meshes and Python code to implement them in FEniCS [17]. In addition, we present numerical results, demonstrating how these algorithms can be applied to both one-dimensional and two-dimensional problems.

This article is organised as follows. Section 2 is devoted to one-dimensional problems, the formulation of finite element methods (FEMs) for them, and the implementation of these methods in FEniCS (with full code presented in Appendix A). In Section 2.2, we show why layer resolving meshes are crucial, and present a particularly important example, the graded Bakhvalov mesh [21]. There are several ways to describe these meshes, but in Section 2.3 we introduce a novel approach, based on MPDEs. The efficient solution of these MPDEs is the topic of Section 2.4. Their effectiveness is verified in Section 2.5 for scalar and coupled systems of equations (in the scalar case, with the code presented in Appendix C).

In Section 3 we consider two-dimensional SPDEs, with a focus on problems for which tensor-product grids are not appropriate. In Section 3.1 we describe two-dimensional adapted meshes. We present MPDEs and an algorithm for generating these meshes in Section 3.2, and numerical results in Section 3.3. FEniCS code for generating two-dimensional layer-adapted meshes using an MPDE is in Appendix D. As such, the contributions of this paper are both expository (motivating MPDEs for mesh generation, and showing how they can be implemented) and novel (presenting a new approach for constructing layer-adapted meshes, and for solving MPDEs for SPDEs).